Higher Several Variable Calculus
Math2111 UNSW

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(Based of Hussain Nawaz's Notes)

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Real one-variable calculus $f : \mathbb{R} \to \mathbb{R}$

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Important Theorems

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- Intermediate Value Theorem
  A continuous function on $[a, b]$ attains all values in $[f(a), f(b)]$.
- Mean Value Theorem
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Mutivariable Calclus Applications $f : \mathbb{R}^n \to \mathbb{R}^m$

- Fluid dynamics
- Black Scholes Options Pricing Model

2 Curves and Surfaces

2.1 Curves

The parameterisation of a curve in $\mathbb{R}^n$ is a vector-valued function

$$\boldsymbol{c} : \boldsymbol{I} \to \mathbb{R}^n$$

where I is an interval on $\mathbb{R}$.

- A multiple point is a point through which the curve passes more than once.
- If $\boldsymbol{I} = [a, b]$ then $\boldsymbol{c}(a)$ and $\boldsymbol{c}(b)$ are called end points.
- A curve is closed if its end points are the same point, $\boldsymbol{c}(a) = \boldsymbol{c}(b)$.

2.2 Limits and Calculus for Curves

For an interval $\boldsymbol{I} \subset \mathbb{R}$ and curve $\boldsymbol{c} : \boldsymbol{I} \to \mathbb{R}^n$ with

$$\boldsymbol{c}(t) = (c_1(t), c_2(t), \ldots c_n(t)),$$

the functions $c_i : \boldsymbol{I} \to \mathbb{R}, i = 1, 2, \ldots, n$ are called the components of $\boldsymbol{c}$.

- If $\lim_{t\to a} c_i(t)$ exists for all i, then $\lim_{t\to a} \boldsymbol{c}(t)$ and

$$\lim_{t\to a} \boldsymbol{c}(t) = \left(\lim_{t\to a} c_1(t), \lim_{t\to a} c_2(t), \ldots \lim_{t\to a} c_n(t)\right)$$

- If $c_i'(t)$ exists for all i, then

$$\boldsymbol{c}'(t) = (c_1'(t), c_2'(t), \ldots, c_n'(t))$$

2.3 Surfaces

You have seen surfaces in $\mathbb{R}^3$ described in 3 ways.

- Graph: $z = f(x, y)$
- Implicitly: $x^2 + y^2 + z^2 = 1$
- Parametrically: $\mathbf{x} = \mathbf{x}_0 + \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2$

3 Analysis

3.1 Formal Definition of a Limit

1-variable Calculus Recall that $\lim_{x\to a} f(x) = L$ requires that for all $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$ then

$$|f(x) - L| < \epsilon.$$

3.2 Distance Functions (metrics)

A function $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which satisfies the following three properties is called a metric.

- **Positive Definite**: for all $x, y \in \mathbb{R}^n$, $d(x, y) > 0$ and $d(x, y) = 0$ iff $x = y$.
- **Symmetric**: for all $x, y \in \mathbb{R}^n$, $d(x, y) = d(y, x)$.
- **Triangle Inequality** for all $x, y, z \in \mathbb{R}^n$, $d(x, y) + d(y, z) \geq d(x, z)$.

Euclidean Distance The Euclidean distance between x and y defined by

$$d(x, y) = ||x - y|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

is a metric.

Equivalent Metrics Two metrics d and δ are considered equal if there exists constants $0 < c < C < \infty$ such that

$$c\delta(x,y) \leq d(x,y) \leq C\delta(x,y).$$

3.3 Limits of Sequences

Ball A ball around $\mathbf{a} \in \mathbb{R}^n$ of radius $\epsilon > 0$ is the set

$$B(\mathbf{a},\epsilon) = \{\mathbf{x} \in \mathbb{R}^n : d(\mathbf{a},\mathbf{x}) < \epsilon\}.$$

Limit of Sequences For a sequence $\{\mathbf{x}_i\}$ of points in $\mathbb{R}^n$ we say that $\mathbf{x}$ is the limit of the sequence if and only if

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies d(\mathbf{x},\mathbf{x}_n) < \epsilon$$

or equivalently

$$\forall \epsilon > 0 \exists N \text{ such that } n \geq N \implies \mathbf{x}_n \in B(\mathbf{x},\epsilon).$$

If $\mathbf{x}$ is the limit of the sequence $\{\mathbf{x}_i\}$ then for each positive ϵ there is a point in the sequence beyond which all points of the sequence are inside $B(\mathbf{x},\epsilon)$.

Convergence

$$\begin{aligned}&\text{A sequence } \mathbf{x}_k \text{ converges to a limit } \mathbf{x}\\ &\quad\Leftrightarrow \text{the components of } \mathbf{x}_k \text{ converge to the components of } \mathbf{x}\\ &\quad\Leftrightarrow d(\mathbf{x}_k,\mathbf{x}) \to 0.\end{aligned}$$

Cauchy Sequences A sequence $\{\mathbf{x}_k\}$ in $\mathbb{R}^n$ is a Cauchy sequence if

$$\forall \epsilon > 0 \exists K \text{ such that } k,l > K \implies d(\mathbf{x}_k,\mathbf{x}_l) < \epsilon.$$

A sequence $\{\mathbf{x}_k\}$ converges in $\mathbb{R}^n$ to a limit if and only if $\{\mathbf{x}_k\}$ is a Cauchy sequence.

3.4 Open and Closed Sets

Definitions Consider x_k

- $x_0 \in \Omega$ is an <u>interior point</u> of Ω if there is a ball around x_0 completely contained in Ω. That is, there exists a $\epsilon > 0$ such that $B(x_0,\epsilon) \subseteq \Omega$.
- Ω is <u>open</u> if every point of Ω is an interior point.
- Ω is <u>closed</u> if its complement is open.
- $x_0 \in \Omega$ is a <u>boundary point</u> of Ω if every ball around x_0 contains points in Ω and points not in Ω.

Closed Sets $\Omega \subset \mathbb{R}^n$ is closed if and only if it contains all of its boundary points.

Union and Intersection

- A finite union/intersection of open sets is open.
- A finite union/intersection of closed sets is closed.

Limit Points and Sets $\mathbf{x}_0$ is a limit point (or accumulation point) of Ω if there is a sequence $\{\mathbf{x}_i\}$ in Ω with limit $\mathbf{x}_0$ and $\mathbf{x}_i \neq \mathbf{x}$.

- Every interior points of Ω is a limit point of Ω.
- $\mathbf{x}_0$ is not necessarily in Ω.
- A set is closed $\Leftrightarrow$ it contains all of its limit points.

Variations of a Set Consider the set $\Omega \in \mathbb{R}^n$.

- The interior of Ω is the set of all its interior points (denoted $\text{Int}(\Omega)$).
- The boundary of Ω is the set of all its boundary points (denoted $\partial\Omega$).
- The closure of Ω is $\Omega \cup \partial\Omega$ (denoted by $\bar{\Omega}$).

The interior is the largest open subset and the closure is the smallest closed set containing Ω.

3.5 Limits

Limit of a Function at a Point Let $\mathbf{b} \in \mathbb{R}^m, \Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \bar{\Omega}$ and let $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. We say that $\mathbf{f}(\mathbf{x})$ converges to $\mathbf{b}$ as $\mathbf{x} \to \mathbf{a}$ if

$$\forall \epsilon > 0 \ \exists \delta > 0 \text{ such that for } \mathbf{x} \in \Omega :$$

$$0 < d(\mathbf{x}, \mathbf{x}_0) < \delta \implies d(\mathbf{f}(\mathbf{x}), \mathbf{b}) < \epsilon.$$

or alternatively

$$\mathbf{x} \in B(\mathbf{a}, \delta) \cap \Omega \implies \mathbf{f}(\mathbf{x}) \in B(\mathbf{b}, \epsilon).$$

If such $\mathbf{b}$ exists, then it is unique and we write

$$\lim_{x \to a} \mathbf{f}(\mathbf{x}) = \mathbf{b}.$$

Useful Limit Theorems Let $\mathbf{b} \in \mathbb{R}^m, \Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \bar{\Omega}$ and let $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. Then

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \iff \lim_{\mathbf{x} \to \mathbf{a}} f_i(\mathbf{x}) = b_i \text{ for all } i = 1, \ldots, m$$

$$\lim_{\mathbf{x} \to \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{b} \iff \lim_{k \to \infty} \mathbf{f}(\mathbf{x}_k) = \mathbf{b}$$

for every sequence $\{\mathbf{x}_k\}_{k=1}^{\infty} \subseteq \Omega$ with $\lim_{k \to \infty} \mathbf{x}_k = \mathbf{a}$.

The first theorem is useful to show that a limit exists whilst the second is useful to show the limit does not exist.

Algebra of limits Given that, $\lim_{x\to x_0} f(x) = a$ and $\lim_{x\to x_0} g(x) = b$, then,

$$\lim_{x\to x_0}(f+g)(x) = a+b$$
$$\lim_{x\to x_0}(fg)(x) = ab$$
$$\lim_{x\to x_0}(\frac{f}{g})(x) = \frac{a}{b}, \text{ given } b \neq 0.$$

Pinching Principle Let $\Omega \subset \mathbb{R}^n$, let $\mathbf{a}$ be a limit point of Ω and let $f, g, h : \Omega \to \mathbb{R}$ be functions such that there exists $\epsilon > 0$ such that

$$g(\mathbf{x}) \leq f(\mathbf{x}) \leq h(\mathbf{x}) \qquad \forall \mathbf{x} \in B(\mathbf{a}, \epsilon) \cap \Omega.$$

Then

$$\lim_{x\to \mathbf{a}} g(\mathbf{x}) = \mathbf{b} = \lim_{x\to \mathbf{a}} h(\mathbf{x}) \implies \lim_{x\to \mathbf{a}} f(\mathbf{x}) = \mathbf{b}.$$

3.6 Continuity

Continuity is like an extension to limits. It first requires that the limit exists and that the limit equals the actual value at that point.

Definition Let $\mathbf{a} \in \Omega \subseteq \mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^m$ be a function. Then f is continuous at $\mathbf{a}$ if and only if

$$\lim_{x\to a} f(\mathbf{x}) = f(\mathbf{a})$$

f is said to be continuous on Ω if it is continuous at $\mathbf{a}$ for every $\mathbf{a} \in \Omega$.

Epsilon-Delta Interpretation

For all $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in B(\mathbf{a}, \delta) \cap \Omega \implies f(x) \in B(f(\mathbf{a}), \epsilon)$.

Continuity by Components All component functions $f_i : \Omega \to \mathbb{R}$ are continuous at $\mathbf{a}$.

Continuity through Sequences For every sequence $\{\mathbf{x}_k\}_{k=1}^{\infty}$ with $\mathbf{x}_k \in \Omega$ for all k, if $\{\mathbf{x}_k\}_{k=1}^{\infty}$ has limit $\mathbf{a}$ then $\{f(\mathbf{x}_k)\}_{k=1}^{\infty}$ converges to $f(a)$.

Elementary Functions If $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ is an elementary function, then f is continuous on Ω.

Preimage Suppose that $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$ is a function. The preimage of a set $U \subseteq \mathbb{R}^m$ is defined by

$$f^{-1}(U) = \{x \in \mathbb{R}^n : f(x) \in U\}.$$

Continuity - Using Preimage Suppose that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$. The following two statements are equivalent.

- f is continuous on Ω.
- $f^{-1}(U)$ is open in $\mathbb{R}^n$ for every open subset U of $\mathbb{R}^m$.

3.7 Path Connected Sets

Definition A set $\Omega \subseteq \mathbb{R}^n$ is said to be path connected if for any $\mathbf{x}, \mathbf{y} \in \Omega$, there is a continuous function φ such that $\varphi(t) \in \Omega$ for all $t \in [0,1]$ and $\varphi(0) = \mathbf{x}$ and $\varphi(1) = \mathbf{y}$.

Theorem Let $\Omega \subseteq \mathbb{R}^n$ and $\mathbf{f} : \Omega \to \mathbb{R}^m$ be continuous. Then

$$B \subseteq \Omega \text{ and } B \text{ path connected} \implies \mathbf{f}(B) \text{ path connected.}$$

3.8 Compact Sets

Bounded A set $\Omega \subseteq \mathbb{R}^n$ is bounded if there is an $M \in \mathbb{R}$ such that $d(\mathbf{x}, \mathbf{0}) \leq M$ for all $\mathbf{x} \in \Omega \iff \Omega \subseteq B(\mathbf{0}, M)$.

Compact A set $\Omega \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.

Theorem Let $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$ be continuous. Then

$$K \subseteq \Omega \text{ and } K \text{ compact} \implies f(K) \text{ compact.}$$

3.9 Bolzano-Weierstrass Theorem

For $\Omega \subseteq \mathbb{R}^n$, the following are equivalent.

1. Ω is compact.
2. Every sequence in Ω has a subsequence that converges to an element of Ω.

4 Differentiation

4.1 Differentiability, Derivatives and Affine Approximations

Differentiability in $\mathbb{R}$ $f : \mathbb{R} \to \mathbb{R}$ is differentiable at some $a \in \mathbb{R}$ means there is a *good* straight-line approximation to f near a called a tangent line. This approximating function is given by

$$T(x) = f(a) + f'(a)(x - a) = f(a) - f'(a)a + f'(a)x = y_0 + L(x)$$

where for all a, $y_0 = f(a) - f'(a)a$ is a fixed number and $L : \mathbb{R} \to \mathbb{R} = f'(a)x$ is the linear map.

Recall that

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Linear Maps A function $L : \mathbb{R}^n \to \mathbb{R}^m$ is called linear iff for all $x, y \in \mathbb{R}^n$ for all $\lambda \in \mathbb{R}$:

$$L(x + y) = L(x) + L(y) \text{ and } L(\lambda x) = \lambda L(x).$$

Affine Maps A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is affine means there is $y_0 \in \mathbb{R}^m$ and a linear map (ie matrix) $\mathbf{L} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\mathbf{x}) = \mathbf{y}_0 + \mathbf{L}(\mathbf{x}).$$

A function $f : \mathbb{R} \to \mathbb{R}$ is affine iff $f(x) = ax + b$, for some $a, b \in \mathbb{R}$.

Affine approximation The function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ has an affine approximation at a point $a \in \Omega$ if and only if there exists a matrix $A \in M_{m\times n}(\mathbb{R})$ such that

$$\lim_{x\to a} \frac{d(f(x) - f(a), A(x - a))}{d(x, a)} = 0$$

If f has an affine approximation at a point $a \in \Omega$, then the matrix A in the definition is called the derivative of f at a and is denoted by $Df(a)$ (or Daf).

The function $T_a f : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T_a f(x) = Df(a)(x - a) + f(a)$$

is called the best affine approximation of f at a.

Differentiability in $\mathbb{R}^n \to \mathbb{R}^n$ A function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ is differentiable for some $a \in \Omega$ if there exists a linear map $L : \mathbb{R}n \to \mathbb{R}^m$ such that

$$\lim_{x\to a} \frac{||f(x) - f(a) - L(x - a)||}{||L(x - a)||} = 0.$$

Notation: the matrix of the linear map L, the derivative of f at a is denoted by $D_a f$.

Delta Epsilon Definition of Differentiability A function $f : \Omega \subset \mathbb{R} \to \mathbb{R}^m$ is differentiable on $a \in \Omega$ if there is a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $\forall\epsilon > 0 \exists\delta > 0$ such that for all $x \in \Omega$

$$||x - a|| < \delta \to ||f(x) - f(a) - L(x - a)|| < \epsilon \, ||x - a|| \, .$$

4.2 Partial Derivatives

Let $\mathbf{a} \in \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function with coordinates x_i and standard basis vectors $\mathbf{e}_i, i \in \{1, \ldots, n\}$. The partial derivative of f in direction i is defined as

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(\mathbf{a} + h\mathbf{e}_i) - f(\mathbf{a})}{h}$$

assuming the limit exists.

Claiaut's Theorem If $f, \dfrac{\partial f}{\partial x_i}, \dfrac{\partial f}{\partial x_j}, \dfrac{\partial^2 f}{\partial x_i x_j}, \dfrac{\partial^2 f}{\partial x_j x_i}$ all exist and are continuous on an open set around $\mathbf{a}$ then

$$\frac{\partial^2 f}{\partial x_i x_j}(\mathbf{a}) = \frac{\partial^2 f}{\partial x_j x_i}(\mathbf{a}).$$

That is the partial derivatives commute.

4.3 Jacobian Matrix

Definition If all partial derivatives of $\mathbf{f} : \Omega \to \mathbb{R}^m$ exists at $\mathbf{a} \in \omega \subseteq \mathbb{R}^n$, then the Jacobian matrix of $\mathbf{f}$ at $\mathbf{a}$ is

$$J_a f = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\mathbf{a}) & \frac{\partial f_1}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_1}{\partial x_n}(\mathbf{a}) \\ \frac{\partial f_2}{\partial x_1}(\mathbf{a}) & \frac{\partial f_2}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_2}{\partial x_n}(\mathbf{a}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\mathbf{a}) & \frac{\partial f_n}{\partial x_2}(\mathbf{a}) & \cdots & \frac{\partial f_m}{\partial x_n}(\mathbf{a}) \end{pmatrix}.$$

Theorem Let $\Omega \subseteq \mathbb{R}^n, \mathbf{a} \in \Omega$ be an interior point and $\mathbf{f} : \Omega \to \mathbb{R}^m$ be a function. If $\mathbf{f}$ is differentiable at $\mathbf{a}$ then all partial derivatives $\dfrac{\partial f_j}{\partial x_i}$ exist at $\mathbf{a}$ and

$$D\mathbf{f}(\mathbf{a}) = J\mathbf{f}(\mathbf{a}).$$

Best affine approximation: $T_a f(x) = Jf(a)(x-a) + f(a)$.

4.4 Differentiable and Continuous

Limit at 0 For $\mathbf{x} \in \mathbb{R}^n$ and L an $m \times n$ matrix,

$$\lim_{x \to \mathbf{0}} ||L\mathbf{x}|| = 0.$$

Open Sets Let $\Omega \in \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^m$ be a function that is differentiable on Ω. Then f is continuous on Ω.

Partial Derivatives + Continuity Let $\Omega \subseteq \mathbb{R}^n$ be open and let $f : \Omega \to \mathbb{R}^m$ be a function. If for all $i = 1, \ldots, n$ and all $j = 1, \ldots, m$ the partial derivative $\dfrac{\partial f_j}{\partial x_i}$ exists and is continuous on Ω then f is differentiable on Ω.

4.5 Chain Rule, Gradient, Directional Derivatives, Tangent Planes

Chain Rule Let $\Omega \subseteq \mathbb{R}^n, \Omega' \subseteq \mathbb{R}^m$ and let $\mathbf{a} \in \Omega$. Suppose $\mathbf{f} : \Omega \to \mathbb{R}^m$ and $\mathbf{g} : \Omega' \to \mathbb{R}^k$ are functions such that $\mathbf{f}(\Omega) \subseteq \Omega'$. If $\mathbf{f}$ is differentiable at $\mathbf{a}$ and $\mathbf{g}$ is differentiable at $\mathbf{f}(\mathbf{a})$, then $\mathbf{g} \circ \mathbf{f}$ is differentiable at $\mathbf{a}$ and

$$D(\mathbf{g} \circ \mathbf{f})(\mathbf{a}) = D\mathbf{g}(\mathbf{f}(\mathbf{a}))D\mathbf{f}(\mathbf{a}).$$

Gradient For $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, if the Jacobian exists, then it is given by the $1 \times n$ matrix

$$Jf = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

This is equivalent to the gradient of f. That is,

$$\text{grad}(f) = \boldsymbol{\nabla} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \cdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}.$$

Directional Derivative The directional derivative of $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ in the direction of the unit vector $\hat{\mathbf{u}}$ at $\mathbf{a} \in \Omega$ is

$$D_{\hat{\mathbf{u}}} f(\mathbf{a}) = f'_{\hat{\mathbf{u}}}(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\hat{\mathbf{u}}) - f(\mathbf{a})}{h}.$$

if the limit exists.

Equivalently, if $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at a then for a unit vector u

$$D_u f(a) = f'_u(a) = \boldsymbol{\nabla} f(a) \cdot u.$$

Alternatively, allowing θ to be the angle between $\boldsymbol{\nabla} f(a)$ and u,

$$D_u f(a) = |\boldsymbol{\nabla} f(a)| \cdot |u| \cdot \cos\theta.$$

Affine Approximation Allow $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ to be a differentiable function at $a \in \Omega$. The best affine approximation to f at a may be written in terms of the gradient vector as

$$T(x) = f(a) + \boldsymbol{\nabla} f(a) \cdot (x - a).$$

Tangent Planes The tangent plane to a function $z = f(x, y)$ is given by

$$z = T(x, y).$$

4.6 Taylor Series and Theorem

Taylor's Theorem For all continuous and differentiable functions $f : \mathbb{R} \to \mathbb{R}$,

$$f(x) \approx P_{k,a}(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n + R_{k,a}(x)$$

where the remainder R is

$$R_{k,a}(x) = \frac{f^{(k+1)}(z)}{(k+1)!}(x-a)^{k+1}$$

for some z between x and a.

$P_{0,a}, P_{1,a}, P_{2,a}, P_{3,a}$ are the best constant, affine, quadratic, cubic approximations.

Hessian Matrix For $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$, the *Hessain matrix of f at a point $a \in \Omega$* is the $n \times n$ matrix

$$Hf(a) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(a) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(a) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(a) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(a) & \frac{\partial^2 f}{\partial x_2^2}(a) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(a) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(a) & \frac{\partial^2 f}{\partial x_2 \partial x_n}(a) & \cdots & \frac{\partial^2 f}{\partial x_n^2}(a) \end{pmatrix}.$$

assuming the 2^{nd} order partial derivatives exist.

Class A function $f : \Omega \to \mathbb{R}, \Omega \subseteq \mathbb{R}^n$ open, is called (of class) C^r if all partial derivatives of f of order $\leq r$ exist and are continuous.

Taylor Polynomials Let $\Omega \subseteq \mathbb{R}^n$ be open, let $a \in \Omega$, and let $f : \Omega \to \mathbb{R}$ be a function of class C^2. The polynomial

$$P_{1,a}(x) = f(a) + \nabla f(a) \cdot (x-a)$$

is called the Taylor polynomial of order 1 about a and the polynomial

$$P_{2,a}(x) = f(a) + \nabla f(a) \cdot (x-a) + \frac{1}{2}(x-a) \cdot Hf(a)(x-a)$$

is called the Taylor Polynomial of order 2 about a.

In general, if $f : \Omega \to \mathbb{R}$ is C^r, Ω open $, a \in \Omega$:

$$P_{r,a}(x) = f(a) + \nabla f(a) \cdot (x-a) + \frac{1}{2}(x-a) \cdot Hf(a)(x-a) \\ + \cdots + \frac{1}{r!} \sum_{i_1,\ldots,i_r=1}^{n} \frac{\partial^r f}{\partial x_{i_1} \ldots \partial x_{i_r}}(a)(x_{i_1} - a_{i_1}) \cdot \cdots \cdot (x_{i_r} - a_{i_r}).$$

Taylor's Theorem (1st order) Let $\Omega \in \mathbb{R}^n$ be open, let $f : \Omega \to \mathbb{R}$ be a function of class C^2. Let $x, a \in \Omega$ s.t. the line segment between x and a is contained in Ω. Then there exist z on this line segment such that

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + R_{1,a}(x)$$

where $R_{1,a}(x) = \frac{1}{2}(x - a) \cdot (Hf(z)(z - a))$.

Taylor's Theorem (2 nd order) Let $\Omega \in \mathbb{R}^n$ be open, let $f : \Omega \to \mathbb{R}$ be a function of class C^3. Let $x, a \in \Omega$ s.t. the line segment between x and a is contained in Ω. Then there exist z on this line segment such that

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + \frac{1}{2}(x - a)Hf(a)(x - a) + R_{2,a}(x)$$

where $R_{2,a}(x) : \Omega \to \mathbb{R}$ is a function such that $\frac{|R_{2,a}(x)|}{|x-a|^2} \to 0$ as $x \to a$.

4.7 Maxima, Minima and Saddle Points

Definitions Let $a \in \Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function. Then

- a is an absolute or *global maximum* of f if $f(a) \geq f(x)$ for all $x \in \Omega$.
- a is an absolute or *global minimum* of f if $f(a) \leq f(x)$ for all $x \in \Omega$.
- a is a *local maximum* of f if there is an open $A \subseteq \Omega$ containing a such that $f(a) \geq f(x)$ for all $x \in A$.
- a is a *local minimum* of f if there is an open $A \subseteq \Omega$ containing a such that $f(a) \leq f(x)$ for all $x \in A$.
- a is a *stationary point* of f if f is differentiable at a and $\nabla f(a) = 0$.
- a is a *saddle point* of f if a is a stationary point of f but it's neither a local max nor a local minimum of f.

Critical Points Let $a \in \Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$ be a function. If a is a local maximum or a local minimum then

1. a is a stationary, or
2. $a \in \partial\Omega \iff a$ is a boundary pt, or
3. f is not differentiable at a.

Points satisfying $1, 2$ or 3 are called critical points.

4.8 Classification of Stationary Points

Definition: An $n \times n$ martix H is

- positive definite $\iff$ all eigenvalues are > 0
- positive semi-definite $\iff$ all eigenvalues are ≥ 0
- positive definite $\iff$ all eigenvalues are < 0
- positive semi-definite $\iff$ all eigenvalues are ≤ 0

Criterion for Local Extrema Let $\Omega \subseteq \mathbb{R}^n$ be open, $a \in \Omega$ and let $f : \Omega \to \mathbb{R}$ be a function such that all paritial derivaitves of f of order at most 2 exists on Ω and $\boldsymbol{\nabla} f(a) = 0$. Then

- $Hf(a)$ is positive definite $\implies$ f has a local minimum at a;
- $Hf(a)$ is negative definite $\implies$ f has a local maximum at a;
- f has a local minimum at $a \implies Hf(a)$ is positive semi-definite;
- f has a local maximum at $a \implies Hf(a)$ is negative semi-definite;

Sylvesetr's Criterion If H_k is the upper $k \times k$ matrix of H and $\Delta_k = det(H_k)$, then

- H is positive definite $\iff$ $\Delta_k > 0$ for all k
- H is positive semi-definite $\implies$ $\Delta_k \geq 0$ for all k
- H is negative definite $\iff$ $\Delta_k < 0$ for all odd k and $\Delta_k > 0$ for all even k
- H is negative semi-definite $\implies$ $\Delta_k \leq 0$ for all odd k and $\Delta_k \geq 0$ for all even k

4.9 Lagrange Multipliers, Implicit and Inverse Function Theorems

Lagrange Multipliers Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ are differentiable and $S = \{x \in \mathbb{R}^n : \varphi(x) = c\}$ defines a smooth surface on $\mathbb{R}^n$. If f attains a local maximum or minimum at a point $a \in S$ then $\boldsymbol{\nabla} f(a)$ and $\boldsymbol{\nabla} \varphi(a)$ are parallel. If $\boldsymbol{\nabla} \varphi(a) \neq 0$, there exist a Lagrange multiplier $\lambda \in \mathbb{R}$ such that

$$\boldsymbol{\nabla} f(a) = \lambda \boldsymbol{\nabla} \varphi(a).$$

Inverse Function Theorem for $f : \mathbb{R} \to \mathbb{R}$ If $f : \mathbb{R} \to \mathbb{R}$ is differentiable on an open interval $I \in \mathbb{R}$ and $f'(x) \neq 0$ for all $x \in I$, then f is invertible on I and the inverse $f^{-1} : f(I) \to \mathbb{R}$ is differentiable with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Generalising the Inverse Function Theorem Let $\Omega \subseteq \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}^n$ be C^1 and suppose $a \in \Omega$. If $Df(a)$ is invertible (as a matrix) then f is invertible on an open set U containing a. That is,

$$f^{-1} : f(U) \to U$$

exists. Furthermore, f^{-1} is C^1 and for $x \in U$,

$$D_{f(x)} f^{-1} = (D_x f)^{-1}.$$

5 Integration

5.1 Riemann Integral

Riemann Integral For a bounded function $f : R \to \mathbb{R}$, if there exists a unique numder I such that

$$\underline{\mathcal{S}}_{\mathcal{P}_1,\mathcal{P}_2}(f) \leq I \leq \overline{\mathcal{S}}_{\mathcal{P}_1,\mathcal{P}_2}(f)$$

for every pair of partitions $\mathcal{P}_1, \mathcal{P}_2$ of R, then f is Riemann integrable on R and

$$I = \iint_R f = \iint_R f(x,y)dA.$$

I is called the Riemann integral of f over R.

Properties of the Riemann Integral For a function of one variable, the Riemann integral is interpreted as the (signed) area bounded by the graph $y = f(x)$ and the x-axis over the interval $[a,b]$. For a function of two variables $\iint_R f$ is the (signed) volume bounded by the graph $z = f(x,y)$ and the xy-plane over the rectangle R. If f and g are integrable on R,

- Linearity: $\iint_R \alpha f + \beta g = \alpha \iint_R f + \beta \int\int_R g, \qquad \alpha, \beta \in \mathbb{R}.$
- Positivity (monotonicity): If $f(x) \leq g(x), \forall x \in R$ then $\iint_R f \leq \iint_R g$
- $\left|\iint_R f\right| \leq \iint_R |f|$
- If $R = R_1 \cup R_2$ and (interior R_1) $\cap$ (interior R_2) $= \emptyset$ then

$$\iint_R f = \iint_{R_1} f + \iint_{R_2} f.$$

5.2 Fubini's Theorem

Fubini's Theorem - Rectangles Let $f : R \to \mathbb{R}$ be continuous on a rectangular domain $R = [a,b] \times [c,d]$. Then f is a bounded function and is integerable over R. Moreover,

$$\int_a^b \int_c^d f(x,y)\,dydx = \int_c^d \int_a^b f(x,y)\,dxdy = \iint_R f.$$

Fubini's Theorem - Discontinuous Let $f : R \to \mathbb{R}$ be bounded on a rectangular domain $R = [a, b]\times[c, d]$ with the discontinuities of f confined to a finite union of graphs of continuous functions. If the integral $\int_c^d f(x, y)\, dy$ exists for each $x \in [a, b]$ then

$$\iint_R f = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx.$$

Similarly, if the integral $\int_a^b f(x, y)\, dx$ exists for each $y \in [c, d]$, then

$$\iint_R f = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Iterated Integrals for Elementary Regions Suppose D is a y-simple region bounded by $x = a, x = b, y = \varphi_1(x)$ and $y = \varphi_2(x)$ and $f : D \to \mathbb{R}$ is continuous. Then

$$\iint_D f = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dxdy.$$

A simile result holds for integrals over x-simple regions.

5.3 Leibniz' Rule

Basic Version Let $a, b, c, d \in \mathbb{R}$ If $f : [a, b] \times [c, d] \to \mathbb{R}$ and $\frac{\partial f}{\partial x}$ are continuous on the rectangle $[a, b] \times [c, d]$. Then

$$g(x) = \int_c^d f(x, y)\, dy.$$

is differentiable and has derivative

$$g'(x) = \frac{d}{dx} \left[\int_c^d f(x, y)\, dy \right] = \int_c^d \frac{\partial f}{\partial x}(x, y) dy \quad \text{for } a \le x \le b.$$

With variable limits Let $a, b \in \mathbb{R}$ with $a \le b$, let $\varphi_1, \varphi_2 : [a, b] \to \mathbb{R}$ be continuously differientiable functions such that $\varphi_1(x) \le \varphi_2(x)$ for all $x \in [a, b]$. If $f : D_1 \to \mathbb{R}$ and $\frac{\partial f}{\partial x}$ are continuous on the region D_1 with

$$D_1 = \{(x, y) : x \in [a, b] \text{ and } \varphi_1(x) \le y \le \varphi_2(x)\}$$

then the function $g(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x, y)\, dy$ is differntiable and

$$g'(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial f}{\partial x}(x, y)\, dy + f(x, \varphi_2(x))\varphi_2'(x) - f(x, \varphi_1(x))\varphi_1'(x).$$

Note: If $\varphi_1(x) \equiv c, \varphi_2(x) \equiv d$ where c, d are constants. Then $g'(x) = \int_c^d \frac{\partial f}{\partial x}\, dy$ (reduced to the previous version).

5.4 Change of Variable

Let $\Omega \subseteq \mathbb{R}^n$ and $F : \Omega \to \mathbb{R}^n$ be an injective and continuously differentiable function such that $\det JF(x)) \neq 0$ for all $x \in \Omega$. If f is any function that is integrable on $\Omega' = F(\Omega)$ then

$$\iint_{\Omega'} (f \circ F)|\det JF|.$$

6 Fourier Series

Fourier Series A Fourier series is the approximation of simple periodic functions by the sum of period functions of the form $\sin(x), \cos(x)$. Note that unlike Taylor series, a function f may be discontinuous. However, any lack of continuity leads to an infinite sum in the Fourier series.

6.1 Inner Products and Norms

Inner Products Let V be a (real) vector space. An inner product on V is a map that assigns each $f, g \in V$ a real number $\langle f, g\rangle$ in such a way that

- $\langle f, f\rangle \geq 0$,
- $\langle f, f\rangle = 0$ if and only if f is zero,
- $\langle \lambda f + \mu g, h\rangle, = \lambda\langle f, h\rangle + \mu\langle g, h\rangle$,
- $\langle g, f\rangle = \langle f, g\rangle$.

for all functions $f, g, h \in V$ and all real constants λ, μ.

Usual Inner Products

- The vector space $\mathbb{R}^n$ consisting of all n-dimensional vector admits the following inner product
$$\langle v, w\rangle = v \cdot w = \sum_{i=1}^{n} v_i w_i.$$
- The vector space $C[a, b]$ consisting of all continuous function defined on the interval $[a, b]$ admits the following inner product
$$\langle f, g\rangle = \int_a^b f(x)g(x)\, dx.$$

Norms A norm on V is a map that assigns each $f \in V$ a real number $||f||$ in such a way that

- $||f|| > 0$,
- $||f|| = 0$ if and only if $f = 0$,
- $||\lambda f|| = |\lambda| \, ||f||$,
- $||f + g|| \leq ||f|| + ||g||$ (triangle inequality)

for all functions $f, g \in V$ and all real constant λ.

Usual Norms Consider a vector space $C[a, b]$ consisting of all continuous functions on $[a, b]$.

- The 2-norm (L^2-norm) is a norm on $C[a, b]$:

$$||f||_2 = \sqrt{\int_a^b f(x)^2 \, dx}$$

- The max norm is a norm on $C[a, b]$:

$$||f||_\infty = \max_{a \leq x \leq b} \{|f(x)|\}$$

Theorem Every inner product on a vector space V induces a norm given by

$$||f|| = \sqrt{\langle f, f \rangle},$$

and the Cauchy-Schwartz inequality holds:

$$|\langle f, g \rangle| \leq ||f|| \, ||g|| \text{ for all } f, g \in V.$$

6.2 Fourier Coefficients and Fourier Series

Fourier Series Suppose that a given function $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic and is square integrable (i.e., $\int_{-\pi}^{\pi} f(x)^2 \, dx < \infty$). Its Fourier series is given by

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} [a_k \cos(kx) + b_k \sin(kx)]$$

where

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) \, dx, \quad k = 0, 1, 2, \ldots$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) \, dx, \quad k = 1, 2, \ldots$$

6.3 Pointwise Convergence of Fourier Series

Piecewise Continuous Functions Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$. Suppose that the one-sided limits $f(c^+) = \lim_{x\to c^+} f(x)$ and $f(c^-) = \lim_{x\to c^-} f(x)$ exists.

- If $f(c^+) = f(c^-) = f(c)$, then f is continuous at c.
- If $f(c^+) = f(c^-) \neq f(c)$ or if $f(c^+) = f(c^-)$ but $f(c)$ is undefined, then f has a removable discontinuity at c.
- If $f(c^+) \neq f(c^-)$, then f has a jump discontinuity at c.

A function $f : [a, b] \to \mathbb{R}$ is piecewise continuous on $[a, b]$ if and only if

(1) For each $x \in [a, b), f(x^+)$ exists;

(2) For each $x \in (a, b], f(x^-)$ exists;

(3) f is continuous on (a, b) except at (most) a finite number of points.

Note that if f is only piecewise continuous then the partial sum of the Fourier series does not necessarily converge to f for all x.

Piecewise Differentiable Functions Consider a function $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$. We write

$$D^+f(c) = \lim_{h\to 0^+} \frac{f(c+h) - f(c^+)}{h}$$

if this one-sided limit exists. Likewise,

$$D^-f(c) = \lim_{h\to 0^-} \frac{f(c+h) - f(c^-)}{h}.$$

A function f is differentiable at c if and only if $f(c^+) = f(c) = f(c^-)$ and $D^+f(c) = D^-f(c)$. A function f is piecewise differentiable on $[a, b]$ if and only if

(1) For each $x \in [a, b), D^+f(x)$ exists;

(2) For each $x \in (a, b], D^-f(x)$ exists;

(3) f is differentiable on (a, b) except at (most) a finite number of points.

Pointwise Convergence Let $c \in \mathbb{R}$ and suppose that a function $f : \mathbb{R} \to \mathbb{R}$ has the following properties:

1. f is 2π-periodic;
2. f is piecewise continuous on $[-\pi, \pi]$;
3. $D^+f(c)$ and $D^-f(c)$ exists.

If f is continuous at c then,

$$S_f(c) = f(c).$$

If f has a jump/removable discontinuity at c, then

$$S_f(c) = \frac{1}{2}[f(c^+) + f(c^-)].$$

6.4 General Periodic, Half Range + Odd and Even Functions

General Periodic Functions Suppose that f has period $2L$, instead of 2π:

$$f(x + 2L) = f(x) \text{ for } x \in \mathbb{R}.$$

Note that $\cos\left(\frac{\pi}{L}x\right)$ and $\sin\left(\frac{\pi}{L}x\right)$ are periodic functions with period $2L$. So, the decomposition becomes

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos\left(\frac{k\pi}{L}x\right) + b_k \sin\left(\frac{k\pi}{L}x\right) \right)$$

where

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi x}{L}\right) \, dx, \quad k = 0, 1, 2, \ldots$$

and

$$b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi x}{L}\right) \, dx, \quad k = 1, 2, \ldots$$

Half Range Expansion Let f be defined on $[0, L]$. We can extend f to an even function (or odd function) on $[-L, L]$ and calculate its Fourier Series.

Odd and Even Functions We define an odd and even functions by the conditions $f(-x) = -f(x)$ and $f(-x) = f(x)$ respectively for a function f. The following elementary properties hold:

- Odd $\times$ Even = Odd
- Odd $\times$ Odd = Even
- Even $\times$ Even = Even
- $\int_{-L}^{L}$ Odd = 0

Odd and Even Functions for Fourier Series If f is odd, then

$$a_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{k\pi x}{L}\right) \, dx = 0$$

and

$$b_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{k\pi x}{L}\right) \, dx = \frac{2}{L} \int_{0}^{L} f(x) \sin\left(\frac{k\pi x}{L}\right) \, dx.$$

So the Fourier series becomes

$$S_f(x) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi x}{L}\right). \qquad \text{(Fourier Sine Series)}$$

If f is even, then

$$a_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{k\pi x}{L}\right)\,dx.$$

and

$$b_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{k\pi x}{L}\right)\,dx = 0$$

So the Fourier series becomes

$$S_f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{k\pi x}{L}\right). \qquad \text{(Fourier Cosine Series)}$$

6.5 Convergence of Sequences

Pointwise Convergence Let $f_k : \mathbb{R} \to \mathbb{R}$. We say f_k converges to f on $[a, b]$ pointwisely iff, for every $x \in [a, b]$, $f_k(x) \to f(x)$ as $k \to \infty$. In this case, f is called the pointwise limit. In terms of $\epsilon - \delta$ language:

For every $x \in [a, b], \epsilon > 0$, there exists an K (depends on ϵ and x), such that

$$|f_k(x) - f(x)| \leq \epsilon \text{ for all } k \geq K.$$

Uniform Convergence Let $f_k : \mathbb{R} \to \mathbb{R}$. We say f_k converges to f on $[a, b]$ uniformly iff for every $\epsilon > 0$, there exists an K (depends on ϵ only), such that

$$\sup_{x\in[a,b]} |f_k(x) - f(x)| \leq \epsilon \text{ for all } k \geq K.$$

Uniform Convergence Theorem If $f_k : \mathbb{R} \to \mathbb{R}$ is continuous on $[a, b]$ for all k if:

- $f_k \to f$ uniformly on $[a, b]$ then f is continuous on $[a, b]$.
- f has at least one discontinuity on $[a, b]$, f_k cannot converge uniformly to f on $[a, b]$.

Weierstrass Test Let $f_k : \mathbb{R} \to \mathbb{R}$ be a sequence of function defined on $[a, b]$. Suppose that there exists a sequence of numbers c_k such that

$$|f_k(x)| \leq c_k \text{ for all } x \in [a, b]$$

and $\sum_{k=1}^{\infty} c_k$ converges (or exists as a real number). Then $\sum_{k=1}^{\infty} f_k$ converges uniformly to a function f on $[a, b]$.

Note that this test also holds for functions $f : \mathbb{R}^n \to \mathbb{R}$ for $x \in \Omega$ where Ω is a closed bounded set in $\mathbb{R}^n$.

Norm Convergence Consider the supremum norm $||f|| = \sup_{x\in[a,b]} |f(x)|$. The definition of uniform convergence can be equivalently written as: for every $\epsilon > 0$, there exists an K such that

$$||f_k - f|| \leq \epsilon \text{ for all } k \geq K.$$

Equivalently,

$$\lim_{k\to\infty} ||f_k - f|| = 0.$$

Here, the norm is defined as the supremum norm. Extending this idea, we can define norm convergence for any arbitrary norm.

Let V be a vector space of functions f equipeed with a norm $||f||$. We say a sequence of functions $f_1, \dots f_k, \dots,$ (norm) converges to f in V if $f \in V$ and

$$\lim_{k\to\infty} ||f_k - f|| = 0.$$

As such, the L^2 norm convergence, also known as mean square convergence is equivalent to the following

$$\lim_{k\to\infty} \int_a^b [f_k(x) - f(x)]^2 dx = 0.$$

Parseval Theorem Let f be 2π periodic, bounded and $\int_{-\pi}^{\pi} f(x)^2\, dx < +\infty$. Then, the Fourier series of f converges to f in the mean square sense. Moreover, the following Parseval's identity holds:

$$\int_{-\pi}^{\pi} f^2(x)\, dx = ||f||_2^2 = \frac{\pi}{2} a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

This identity continues to hold for $2L$ periodic functions integrated over $[-L, L]$.

7 Vector Fields

7.1 Vector Fields and Flow

Vector Fields A vector field in 3D space has components that are functions and is of the type

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= \mathbf{F}(x, y, z) \\ &= (F_1(x, y, z), F_2(x, y, z), F_3(x, y, z)) \\ &= F_1(x, y, z)\mathbf{i} + F_2(x, y, z)\mathbf{j} + F_3(x, y, z)\mathbf{k}.\end{aligned}$$

A vector field in 2D has components that are functions and is of the type

$$\begin{aligned}\mathbf{F}(\mathbf{x}) &= \mathbf{F}(x, y) \\ &= (F_1(x, y), F_2(x, y)) \\ &= F_1(x, y)\mathbf{i} + F_2(x, y)\mathbf{j}.\end{aligned}$$

Flow Lines If $\mathbf{F}$ is a vector field, a *flow line* for $\mathbf{F}$ is a path $\mathbf{c}(t)$ such that

$$\mathbf{c}'(t) = \mathbf{F}(\mathbf{c}(t)).$$

That is, $\mathbf{F}$ yields the velocity field of the path $\mathbf{c}(t)$.

The Del $\boldsymbol{\nabla}$ operator The vector differential operator $\boldsymbol{\nabla}$ is not a vector, but an operator. It may be considered a symbolic vector. The differential operator may be written as

$$\boldsymbol{\nabla} = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Divergence If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the divergence of $\mathbf{F}$ is the scalar field

$$\text{div } \mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Divergence may be thought as a type of derivative that describes the measure at which a vector field *spreads away* from a certain point. If the divergence is positive, then there is a net outflow while there is net inflow if the divergence is negative.

Observe that the divergence of a vector field will be real-valued.

Curl If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the curl of $\mathbf{F}$ is the vector field

$$\begin{aligned} \text{curl } \mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \end{aligned}$$

Curl is also analogous to a type of derivative for vector fields. The curl may be thought as the measure at which the vector field *swirls* around a point. A positive swirl can be thought of as a counterclockwise rotation.

Observe that the curl of a vector field is also a vector field.

7.2 Vector Identities

Basic Vector Identities

1. $\boldsymbol{\nabla}(f + g) = \boldsymbol{\nabla} f + \boldsymbol{\nabla} g$
2. $\boldsymbol{\nabla}(\lambda f) = \lambda \boldsymbol{\nabla} f$ where $\lambda \in \mathbb{R}$
3. $\boldsymbol{\nabla}(fg) = f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f$. You may draw analogies to the product.
4. $\boldsymbol{\nabla}\left(\frac{f}{g}\right) = \frac{g\boldsymbol{\nabla} f - f\boldsymbol{\nabla} g}{g^2}$ where $g \neq 0$. This is analogous to the quotient rule.

5. $\boldsymbol{\nabla} \cdot (\mathbf{F} + \mathbf{G}) = \boldsymbol{\nabla} \cdot \mathbf{F} + \boldsymbol{\nabla} \cdot \mathbf{G}$

6. $\boldsymbol{\nabla} \times (\mathbf{F} + \mathbf{G}) = \boldsymbol{\nabla} \times \mathbf{F} + \boldsymbol{\nabla} \times \mathbf{G}$

7. $\boldsymbol{\nabla} \cdot (f\mathbf{F}) = f\boldsymbol{\nabla} \cdot \mathbf{F} + \mathbf{F} \cdot \boldsymbol{\nabla} f$

8. $\boldsymbol{\nabla} \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) - \mathbf{F} \cdot (\boldsymbol{\nabla} \times \mathbf{G})$

9. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \mathbf{F}) = 0$

10. $\boldsymbol{\nabla} \times (f\mathbf{F}) = f\boldsymbol{\nabla} \times \mathbf{F} = \boldsymbol{\nabla} f \times \mathbf{F}$

11. $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} f) = 0$

12. $\boldsymbol{\nabla}^2(fg) = f\boldsymbol{\nabla}^2 g + 2(\boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g) + g\boldsymbol{\nabla}^2 f$

13. $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f \times \boldsymbol{\nabla} g) = 0$

14. $\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) = f\boldsymbol{\nabla}^2 g - g\boldsymbol{\nabla} f^2$

8 Path Integrals

8.1 Path Integrals

Path (scalar line) Integrals We say that a vector-valued function $\mathbf{c}(t)$ parametrises a curve C for $a < t < b$ if the image of $\mathbf{c}$ traces out the curve C.

Computing a Scalar Line Integral Let $\mathbf{c}(t)$ be a parametrisation of a curve $C \in \mathbb{R}^3$ for $a < t < b$. Assume that $f(x, y, z)$ and $\mathbf{c}'(t)$ are continuous. Then

$$\int_C f(x, y, z)\, ds = \int_a^b f(\mathbf{c}(t))||\mathbf{c}'(t)||\, dt$$

The value of the integral on the right does not depend on the choice of parametrisation. For $f(x, y, z) = 1$, we obtain the length of C:

$$\text{Length of } C = \int_C ||\mathbf{c}'(t)||\, dt$$

where $||\mathbf{c}'(t)|| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}$ for $\mathbf{c}(t) = (x(t), y(t), z(t))$.

Elementary Properties of Path Integral

- $\int_C f_1\, ds + \int_C f_2\, ds = \int_C (f_1 + f_2)\, ds$
- $\int_C \lambda f\, ds = \lambda \int_C f\, ds, \quad \lambda \in \mathbb{R}$

8.2 Applications of Path Integrals

Mass Suppose that $\delta = \delta(x, y, z)$ which is a density function.

$$M = \int_C \delta(x, y, z)\, dz$$

First Moments About the Coordinate Planes

$$M_{yz} = \int_C x\delta\, ds, \qquad M_{xz} = \int_C y\delta\, ds, \qquad M_{xy} = \int_C z\delta\, ds$$

Coordinates of the Center of Mass

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

Moments of Inertia about Axes

$$I_x = \int_C (y^2 + z^2)\delta\, dx, \qquad I_y = \int_C (x^2 + z^2)\delta\, ds, \qquad I_z = \int_C (x^2 + y^2)\delta\, ds$$

9 Vector Line Integrals

9.1 Vector Line Integrals

Vector Line Integrals There is an important distinction between vector and scalar line integrals. To define a vector line integral we must specify a direction along the path or curve C.

A curve C can be traversed in one of two directions. We say that C is oriented if one of these two directions is specified. We refer to the specified direction as the forward direction along the curve.

Computing a Line Integral Let $\mathbf{c}(t)$ be a parameterisation of an oriented curve C for $a \leq t \leq b$. The line integral of a vector field $\mathbf{F}$ along C is the defined by

$$\int_C \mathbf{F} \cdot ds = \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt.$$

Link with the path integral Let $\mathbf{c}(t)$ be a parametrisation of an oriented smooth curve C and let $\hat{\mathbf{T}}$ denotes the unit tangent vector pointing in the forward direction of C.

$$\hat{\mathbf{T}}(\mathbf{c}(t)) = \frac{\mathbf{c}'(t)}{||\mathbf{c}'(t)||}$$

Then, the line integral of a vector field $\mathbf{F}$ over the oriented curve C is the path integral of the tangential component of $\mathbf{F}$ along C, that is

$$\int_c \mathbf{F} \cdot ds = \int_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds.$$

Summing Paths Let $C_i, i = 1, \dots, m$ be curves with continuous differentiable parameterisations. Let $C = C_1 + C_2 + \cdots + C_m$, that is, C is the union of curves C_i, which are joined end-to-end. Then, we define

$$\int_C \mathbf{F} \cdot ds = \sum_{i=1}^{m} \int_{C_i} \mathbf{F} \cdot ds.$$

Work notation Denote $\mathbf{c}(t) = (x(t), y(t), z(t))$ and $\mathbf{F} = (M, N, P) = M\mathbf{i}, N\mathbf{j}, P\mathbf{k}$. Then, we can denote work as any of the following notations:

$$\begin{aligned}
W &= \int_C \mathbf{F} \cdot ds \\
&= \int_a^b \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)\, dt && \text{(Definition)} \\
&= \int_a^b \left(M\frac{dx}{dt} + N\frac{dy}{dt} + P\frac{dz}{dt} \right) dt \\
&= \int_a^b M dx + N dy + P dz. && \text{(Alternative form)}
\end{aligned}$$

Properties of Line Integrals Let C be a smooth oriented curve and let $\mathbf{F}$ and $\mathbf{G}$ be vector fields.

(i) Linearity:

$$\int_C (\mathbf{F} + \mathbf{G}) \cdot ds = \int_C \mathbf{F} \cdot ds + \int_C \mathbf{G} \cdot ds$$
$$\int_C k\mathbf{F} \cdot ds = k \int_C \mathbf{F} \cdot ds \qquad (k \text{ a constant})$$

(ii) Reversing orientation:

$$\int_{-C} \mathbf{F} \cdot ds = -\int_C \mathbf{F} \cdot ds$$

(iii) Additivity: If C is a union of n smooth curves $C_1 + \cdots + C_n$, then

$$\int_C \mathbf{F} \cdot ds = \int_{C_1} + \cdots + \int_{C_n} \mathbf{F} \cdot ds$$

9.2 Other Applications

Flow Integral, Circulation If $\mathbf{r}(t)$ is a smooth curve in the domain of a continuous velocity field $\mathbf{F}$, the flow along the curve from $t = a$ to $t = b$ is

$$\text{Flow} = \int_a^b \mathbf{F} \cdot \hat{\mathbf{T}}\, ds$$

The integral in this case is called a flow integral. If the curve is a closed loop, the flow is called the circulation around the curve.

Flux Across a Closed Curve in the Plane If C is a smooth closed curve in the domain of a continuous vector field $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ in the plane and if $\hat{\mathbf{n}}$ is the outward-pointing unit normal vector on C, the flux of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds.$$

Calculating Flux Across a Smooth Closed Plane Curve

$$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M\, dy - N\, dx$$

The integral can be evaluated from any smooth parametrisation $x = g(t), y = h(t), a \leq t \leq b$, that traces C counterclockwise exactly once.

9.3 Fundamental Theorem of Line Integrals

(Second) Fundamental Theorem of Calculus in One Vairable Let $f : \mathbb{R} \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$. If $f(x) = \varphi'(x)$, then

$$\int_a^b \varphi'(x)\, dx = \int_a^b f(x)\, dx = \varphi(b) - \varphi(a).$$

Gradient Fields A vector field $\mathbf{F}$ is called a gradient vector field if there exists a real-valued function φ such that $\mathbf{F} = \boldsymbol{\nabla}\varphi$. That is, $(M, N, P) = (\frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z})$. A vector field $\mathbf{F}$ with this property is called conservative and φ is called the potential function of $\mathbf{F}$.

Fundamental Theorem for Gradient Vector Fields If $\mathbf{F} = \boldsymbol{\nabla}\varphi$ on a domain $\mathcal{D}$, then for every oriented smooth curve C in $\mathcal{D}$ with initial point P and terminal point Q.

$$\int_C \mathbf{F} \cdot ds = \varphi(Q) - \varphi(P)$$

If C is closed (i.e., if $P = Q$), then $\oint_C \mathbf{F} \cdot ds = 0$.

Cross Partials of a Gradient Vector Field are Equal Let $\mathbf{F} = (F_1, F_2, F_3)$ be a gradient vector field whose components have continuous partial derivatives. Then the cross partials are equal:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$$

Similarly, if the vector field in the plane $\mathbf{F} = (F_1, F_2)$ is the gradient vector field, then $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$. Equivalently, $\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}$.

9.4 Green's Theorem

Green's Theorem connects double integrals with line integrals and is very useful for line integrals over complicated vector fields with simpler partial derivatives.

Green's Theorem (Flux-divergence or Normal Form) Let D be a bounded simple region in $\mathbb{R}^2$ with nonempty interior, whose boundary consists of a finite number of smooth curves. Let C be the boundary of D with a positive (counter-clockwise) direction. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field which is continuously differentiable on D. Then, the outward flux of $\mathbf{F}$ across the curve C equals the double integral of divergence $\boldsymbol{\nabla} \cdot F$ over D, that is

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{n}}\, ds = \oint_C -N\, dx + M\, dy = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy$$

Three key assumptions:

- The region D is bounded and simple region with nonempty interior.
- The boundary C is oriented in the positive (counter-clockwise) direction, and is a finite union of smooth curves.
- The vector field $\mathbf{F}$ is continuously differentiable on D.

Green's Theorem (Circulation-curl or Tangential Form) Let D be a bounded simple region in $\mathbb{R}^2$ with nonempty interior, whose boundary consists of a finite number of smooth curves. Let C be the boundary of D with a positive (counter-clockwise) direction. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field which is continuously differentiable on D. Then, the counter-clockwise circulation of $\mathbf{F}$ around C equals the double integral $\boldsymbol{\nabla} \times F \cdot k$ over D, that is

$$\oint_C \mathbf{F} \cdot \hat{\mathbf{T}}\, ds = \oint_C M\, dx + N\, dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Area of a Region Let D be a simple and bounded region with non-empty interior and let C be its boundary with positive (counter-clockwise) direction which is a finite union of smooth curves. Then, the area of D can be calculated by

$$\text{Area}(D) = \frac{1}{2} \oint_C (-y\, dx + x\, dy).$$

10 Surface Integrals

10.1 Parametrised Surfaces

Parametrised Surface A parametrised surface is a function $\phi : D \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, where D is some domain in $\mathbb{R}^2$, that is,

$$\phi(u, v) = (x(u, v), y(u, v), z(u, v)).$$

The surface S corresponding to the function ϕ is its image: $S = \phi(D)$. If ϕ is differentiable (resp. continuously differentiable), then we call S a differentiable (resp. continuously differentiable) surface.

Cone The cone $z^2 = x^2 + y^2$ has the parametrisation

$$\phi(u, v) = (u \cos v, u \sin v, u), \qquad 0 \le v \le 2\pi,\ u \in \mathbb{R}.$$

Cylinder The cylinder of radius $R, x^2 + y^2 = R^2$ has the parametrisation

$$\phi(\theta, z) = (R \cos \theta, R \sin \theta, z), \qquad 0 \le \theta \le 2\pi,\ z \in \mathbb{R}.$$

Sphere The sphere of radius $R, x^2 + y^2 + z^2 = R^2$ has the parametrisation

$$\Phi(\theta, \phi) = (R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi), \qquad 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$$

10.2 Surface Area

In the rest of this section, we consider smooth parametrised surfaces and also piecewise smooth parametrised surfaces.

Area of a Surface Let $\Phi(u, v)$ be parametrisation of a smooth surface S with parameter domain D. The area of the surface S is

$$\text{Area}(S) = \iint_D ||\mathbf{T}_u \times \mathbf{T}_v||\, dudv.$$

Sometimes we write

$$||\mathbf{n}(u, v)|| = ||\mathbf{T}_u \times \mathbf{T}_v||.$$

Note that this $\mathbf{n}(u, v)$ is not necessarily a unit vector and neither are the tangent vectors.

10.3 Surface Integral

Let $\Phi(u, v)$ be a parametrisation of a smooth parametrised surface S with parameter domain D. The surface integral of f over S is

$$\begin{aligned} &\iint_S f(x, y, z)\, dS \\ &= \iint_D f(\Phi(u, v))||\mathbf{T_u} \times \mathbf{T_v}||\, dudv \\ &= \iint_D f(\Phi(u, v))||\mathbf{n}(u, v)||\, dudv. \end{aligned}$$

If S is piecewise smooth parameterised surface S which are made up of finitely many smooth surface $S_i, i = 1, \ldots, m$, then, the surface integral of f over S is

$$\iint_S f(x, y, z)\, dS = \sum_{i=1}^{m} \iint_{S_i} f(x, y, z)\, dS.$$

10.4 Surface Integrals of Vector-Valued Functions

The surface integral of a vector field $\mathbf{F}$ over an oriented smooth parametrerised surface S is defined as

$$\iint_S \mathbf{F} \cdot dS = \iint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, dS.$$

More generally, for a piecewise smooth parametrised surface S formed by finite union of oriented smooth surfaces $S_i, i = 1, \ldots, m$, then

$$\iint_S \mathbf{F} \cdot dS = \sum_{i=1}^{m} \iint_{S_i} \mathbf{F} \cdot dS.$$

If S is a smooth parametrised oriented surface and Φ parameterises the surface S (i.e., $\hat{\mathbf{n}}$ in the normal direction specificed by the orientation of S) then,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot dS &= \iint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, dS \\
&= \iint_D \left(\mathbf{F}(\Phi(u,v)) \cdot \frac{\mathbf{T_u} \times \mathbf{T_v}}{||\mathbf{T_u} \times \mathbf{T_v}||} \right) ||\mathbf{T_u} \times \mathbf{T_v}||\, dudv \\
&= \iint_D \mathbf{F}(\Phi(u,v)) \cdot (\mathbf{T_u} \times \mathbf{T_v})\, dudv
\end{aligned}$$

11 Integral Theorems

11.1 Stokes Theorem

Stokes theorem gives the relationship between a surface integral over a surface S and a linear integral around the boundary curve of S.

Let S be a smooth oriented surface defined by a one-to-one parametrisation $\Phi : D \subset R^2 \to S$, where D is a region to which Green's theorem applies. Let ∂S denote the oriented boundary of S and let $\mathbf{F}$ be a C^1 vector field on S. Then

$$\iint_S (\boldsymbol{\nabla} \times F) \cdot dS = \int_{\partial S} F \cdot ds.$$

11.2 (Gauss) Divergence Theorem

The divergence theorem gives the relationship between a triple integral over a region W and a surface integral over its boundary surface S.

Let $W \subseteq \mathbb{R}^3$ be a bounded, solid and simple region, and let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$ which is continuously differentiable on W. Let S be the boundary of W which is a piece-wise smooth parameterised surface formed by a finite union of oriented smooth surfaces (say S_i). Then, the outward flux of $\mathbf{F}$ across the surface S equals the triple integral of divergence div$\mathbf{F}$ over W, that is

$$\iint_S \mathbf{F} \cdot dS = \iiint_W \boldsymbol{\nabla} \cdot \mathbf{F}\, dV$$

where $\iint_S \mathbf{F} \cdot dS = \sum \iint_{S_i} \mathbf{F} \cdot dS$ and the surface are oriented such that the normal vector points outwards.