Higher Theory and Applications of Differential Equations
MATH2221 UNSW

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Contents

Chapter 1

Linear ODEs

1.1 Introduction

Recall that a first-order ordinary differential equation (ODE) has, in its most general realisation, the form

$$y'(t) = f(t, y(t)).$$

A special case is the equation

$$a(t)y'(t) + b(t)y(t) = f(t),$$

with $a(t) \neq 0$ on some interval $I \in \mathbb{R}$. This special first-order ODE is called a **linear first-order ODE**. Another special case is

$$y'(t) = f(t)g(y),$$

which is known as a **separable first-order ODE**.

For a separable equation the solution is found (at least, implicitly by) writing:

$$\int \frac{1}{g(y)}\, dy = \int f(t)\, dt.$$

Solving Seperable ODEs Consider $y' = t^2 y, y(0) = 3$. This is seperable with $f(t) = t^2$ and $g(y) = y$. Then

$$\int \frac{1}{y}\, dy = \int t^2\, dt$$

so that

$$\ln |y(t)| = \frac{1}{3}t^3 + C.$$

Now apply e^t to both sides to obtain

$$|y(t)| = e^{\frac{1}{3}t^3 + C} = e^C e^{\frac{1}{3}t^3}.$$

Thus, a general solution of the equation is

$$y(t) = Ae^{\frac{1}{3}t^3}.$$

Since $y(0) = 3$, we see that the unique solution is $y(t) = 3e^{\frac{1}{3}t^3}$.

In the case of a linear first-order equation, i.e. $y' + a(t)y = f(t)$, a useful solution method is the integrating factor technique. The idea is to find a function μ so that when we multiply both sides of the equation with μ we find that

$$[\mu y](t)' = \mu(t)(y' + a(t)y) = \mu(t)f(t),$$

for if this happens, then the general solution of the ODE should be

$$y(t) = \frac{1}{\mu(t)} \int \mu(t) f(t)\, dt + \frac{C}{\mu(t)}.$$

Solving Linear First-Order ODE Solve $y' - 2ty = 3t$. We pick

$$\mu(t) = e^{\int -2t\, dt} = e^{-t^2}.$$

Then

$$\begin{aligned} (e^{-t^2} y)' &= 3te^{-t^2} \\ e^{-t^2} y &= \int 3te^{-t^2}\, dt = -\frac{3}{2} e^{-t^2} + C \\ y(t) &= -\frac{3}{2} + Ce^{t^2}. \end{aligned}$$

1.2 Linear Differential Operators

In linear algebra, you have seen the compact notation $A\mathbf{x} = \mathbf{b}$ for system of linear equations. A similar notation when dealing with a linear ordinary differential equations is

$$Lu = f.$$

Here, L is an operator (or transformation) that acts on a function u to create a new function Lu. Given coefficients $a_0(x), a_1(x), \ldots, a_m(x)$ we define the **linear differential operator** L of **order** m,

$$\begin{aligned} Lu(x) &= \sum_{j=0}^{m} a_j(x) D^j u(x) \\ &= a_m D^m u + a_{m-1} D^{m-1} u + \cdots + a_o u, \end{aligned}$$

where $D^j u = d^j u / dx^j$ (with $D^0 u = u$).

We refer to a_m as the **leading coefficient** of L and assume that each $a_j(x)$ is a smooth function of x.

The ODE $Lu = f$ is said to be **singular** with respect to an interval $[a, b]$ if the leading coefficient $a_m(x)$ vanishes for any $x \in [a, b]$.

Example $Lu = (x - 3)u''' - (1 + \cos x)u' + 6u$ is a linear differential of order 3, with leading coefficient $x - 3$. Thus, L is singular on $[1, 4]$, but not singular on $[0, 2]$.

Example $N(u) = u'' + u^2 u' - u$ is a nonlinear differential operator of order 2.

Linearity For any constants c_1 and c_2 and any m-times differentiable functions u_1 and u_2,

$$L(c_1 u_1 + c_2 u_2) = c_1 L u_1 + c_2 L u_2.$$

Ordinary differential equations of the form $Lu = 0$ are known as **homogenous**. Those of the form $Lu = f$ are known as **inhomogeneous**.

When the solution to a differential equation is prescribed at a particular point $x = x_0$, that is

$$u(x_0) = v_0, \quad u'(x_0) = v_1, \quad \dots, \quad u^{(m-1)}(x_0) = v_{m-1},$$

we call it an **initial value problem**. Where a differential equation is order 2 or greater, solutions at 2 or more locations can be prescribed. Such problems are called **boundary value problems**.

Unique Solution to Linear Initial Problem For an ODE $Lu = f$ which is not singular with repsect to a, b, with f continuous on $[a, b]$, the IVP for an mth-order linear differential operator with m inital values has a unique solution.

Solution to mth Order Problem has Dimension m Assume that the linear, mth-order differential operator L is not singular on $[a, b]$. Then the set of all solutions to the homogenous equation $Lu = 0$ on $[a, b]$ is a vector space of dimension m.

If $\{u_1, u_2, \dots, u_m\}$ is **any** basis for the solution space of $Lu = 0$, then every solution can be written in a unique way as

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_m u_m(x) \quad \text{for } a \leq x \leq 4.$$

We refer to this as the **general solution** of the homogenous equation $Lu = 0$ on $[a, b]$.

Linear superposition refers to this technique of constructing a new solution out of a linear combination of old ones.

Example The general solution to $u'' - u' - 2u = 0$ is $u(x) = c_1 e^{-x} + c_2 e^{2x}$.

Consider the inhomogeneous equation $Lu = f$ on $[a, b]$, and fix a particular solution u_P.
For *any* solution u, the difference $u - u_P$ is a solution of the homogenous equation because

$$L(u - u_P) = Lu - Lu_P = f - f = 0 \text{ on } [a, b].$$

Hence, $u(x) - u_P(x) = c_1 u_1(x) + \dots_c m u_m(x)$ for some constants $c_1, \dots, c_m$ and so

$$u(x) = u_P(x) + \underbrace{c_1 u_1(x) + \cdots + c_m u_m(x)}_{u_H(x)}, \quad a \leq x \leq b,$$

is the **general solution** of the inhomogeneous equation $Lu = f$.

Example The inhomogenous ODE $u'' - u' - 2u = -2e^x$ has a particular solution $u_P(x) = e^x$.
The general solution for its homogenous counterpart is $u_H(x) = c_1 e^{-x} + c_2 e^{2x}$.
So the general solution of the inhomogeneous ODE is

$$u(x) = u_P(x) + u_H(x) = e^x + c_1 e^{-x} + c_2 e^{2x}.$$

Reduction of Order For $u = u_1(x) \neq 0$, a solution to the ODE

$$u'' + p(x)u' + q(x)u = 0,$$

on some interval I, then a second solution is

$$u = u_1(x) \int \frac{1}{u_1^2 \exp\left(\int p \, dx\right)} \, dx.$$

Example For the ODE $u'' - 6u' + 9u = 0$, take $u_1 = e^{3x}$ and find v. **Answer** xe^{3x}.

1.3 Differential Operators with Constant Coefficients

If L has constant coefficients, then the problem of solving $Lu = 0$ reduces to that of factorising the polynomial having the same coefficients.

Suppose that a_j is constant for $0 \leq j \leq m$, with $a_m \neq 0$. We define the associated polynomial of degree m,

$$p(z) = \sum_{j=0}^{m} a_j z^j = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0,$$

so that if

$$Lu = a_m u^{(m)} + a_{m-1} u^{(m-1) + \cdots + a_1 u + a_0},$$

then formally, $L = p(D)$.

By the fundamental theorem of algebra,

$$p(z) = a_m (z - \lambda_1)^{k_1} (z - \lambda_2)^{k_2} \cdots (z - \lambda)^{k_r}$$

where $\lambda_1, \lambda_@, \ldots, \lambda_r$ satisfying

$$k_1 + k_2 + \cdots + k_r = m.$$

Lemma $(D - \lambda) x^j e^{\lambda x} = j x^{j-1} e^{\lambda x}$ for $j \geq 0$.

Lemma $(D - \lambda)^k x^j e^{\lambda x} = 0$ for $j = 0, 1, \ldots, k - 1$.

Basic Solutions If $(z - \lambda)^k$ is a factor of $p(z)$ then the function $u(x) = x^j e^{\lambda x}$ is a solution of $Lu = 0$ for $0 \leq j \leq k - 1$.

General Solution For the constant-coefficient case, the general solution of the homogenous equation $Lu = 0$ is

$$u(x) = \sum_{q=1}^{r} \sum_{l=0}^{k_q - 1} c_{ql} x^l e^{\lambda_q x},$$

where the c_{ql} are arbitrary constants.

Repeated Real Root From the factorisation

$$D^4 + 6D^3 + 9D^2 - 4D - 12 = (D-1)(D+2)^2(D+3)$$

we see that the general solution of

$$u'''' + 6u''' + 9'' - 4u' - 12u = 0$$

is

$$u = c_1 e^x + c_2 e^{-2x} + c_3 x e^{-2x} + c_4 e^{-3x}.$$

Complex Root From the factorisation

$$\begin{aligned} D^3 - 7D^2 + 17D - 15 &= (D^2 - 4D + 5)(D-3) \\ &= (D-2-i)(D-2+i)(D-3) \end{aligned}$$

we see that the general solution of

$$u''' - 7u'' + 17u' - 15u = 0$$

is

$$\begin{aligned} u(x) &= c_1 e^{(2+i)x} + c_2 e^{(2-i)x} + c_3 e^{3x} \\ &= c_4 e^{2x} \cos x + c_5 e^{2x} \sin x + c_3 e^{3x}. \end{aligned}$$

Second-order ODEs arise naturally in classical mechanics for example a harmonic simple oscillator.

1.4 Wronskians and Linear Independence

We introduce a function, called the Wronskain that provides us with a way of testing whether a family of solutions to $Lu = 0$ is linearly independent.

Let $u_1(x), u_2(x), \ldots, u_m(x)$ be functions defined on an interval $I \in \mathbb{R}$. The functions $u_1, \ldots, u_m$ are called **linearly dependent** if there exist constant $a_1, a_2, \ldots, a_m$ **not all zero** such that

$$a_1 u_1(x) + a_2 u_2(x) + \cdots + am u_m(x) = 0 \quad \forall x \in I.$$

If the above equation only holds for

$$a_i = 0, \quad i = 1, 2, \ldots, m$$

then the functions are **linearly independent**.

Example $u_1 = \sin 2x$ and $u_2 = \sin x \cos x$ are linearly dependent.
$u_1 = \sin x$ and $u_2 = \cos x$ are linearly indepdent.

The **Wronskian** of the functions $u_1, u_2, \ldots, u_m$ is the $m \times m$ determinant

$$W(x) = W(x; u_1, u_2, \ldots, u_m) = \det\big[D^{i-1} u_j\big].$$

Example The Wronskian of the functions $u_1 = e^{2x}, u_2 = xe^{2x}$ and $u_3 = e^{-x}$ is

$$W = \begin{vmatrix} e^{2x} & xe^{2x} & e^{-x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} & -e^{-x} \\ 4e^{2x} & 4e^{2x} + 4xe^{2x} & e^{-x} \end{vmatrix} = 9e^{3x}.$$

Lemma If $u_1, \ldots, u_m$ are linearly dependent over an interval $[a, b]$ then $W(x; u_1, \ldots, u_m) = 0$ for $a \leq x \leq b$.

Lemma If $u_1, u_2, \ldots, u_m$ are solutions of $Lu = 0$ on the interval $[a, b]$ then their Wronskain satisfies

$$a_m(x)W'(x) + a_{m-1}(x)W(x) = 0, \quad a \leq x \leq b.$$

Linear Independence of Solutions Let $u_1, u_2, \ldots, u_m$ be solutions of a non-singular, linear, homogenous, m-th order ODE $Lu = 0$ on the interval $[a, b]$.
Either
$W(x) = 0$ for $a \leq x \leq b$ and the m solutions are linearly **dependent**,
or else
$W(x) \neq 0$ for $a \leq x \leq b$ and the m solutions are linearly **independent**.

1.5 Methods for Inhomogeneous Equations

1.5.1 Judicious Guessing Method

You would have learned the mthod of undetermined coefficients for constructing a particular solution u_P to an inhomogeneous second-order linear ODE $Lu = f$ in some simple cases. We will study this method systematically for higher-order linear ODEs with constant coefficients.

Superposition of Solutions Suppose that u_1 solves $Lu = e^{3x}$, and u_2 solves $Lu = \sin x$, where L is a linear differential operator. Then the solution of

$$Lu = e^{3x} + \sin x$$

is

$$u(x) = u_1(x) + u_2(x).$$

And a solution of

$$Lu = \frac{1}{2}e^{3x} - 5\sin x$$

is

$$u(x) = \frac{1}{2}u_1(x) - 5u_2(x).$$

Now we want to investigate some methods for finding particular solutions - i.e., finding a solution of $Lu = f$. One such method is the method of judicious guessing. For example:

1. If f is a polynomial, then guess that u_p is a polynomial.

2. If f is a exponential, then guess that u_p is exponential.

3. If f is a sine or cosine, then guess that u_p is a combination of such functions.

One problem with this method: it will only work for the types of functions identified above.

Example Suppose that $u'' - u' = t^2 + 2t$. Note as before that,

$$u_h(t) = c_1 + c_2 e^t.$$

So guess,

$$u_p(t) = At^3 + Bt^2 + Ct + D.$$

Then

$$t^2 + 2t = u_p'' - u' = -3At^2 + (6A - 2B)t + (2B - C).$$

So, equating coefficients of like power terms, we see that

$$A = -\frac{1}{3}, B = -2, C = -4, \text{ and } D \text{ is unrestricted.}$$

Therefore, reabsorbing D into c_1, we see that

$$u(t) = u_h(t) + u_p(t) = c_1 + c_2 e^t - \frac{1}{3}t^3 - 2t^2 - 4t.$$

Now we look at this idea of judicious guessing in a more systematic way. Let $L = p(D)$ be a linear differential operator of order m with constant coefficients.

Polynomial Solutions Assume that $a_0 = p(0) \neq 0$. For any integer $r \geq 0$, there exists a unique polynomial u_P of degree r such that $Lu_P = x^r$.

Exponential Solutions Let $L = p(D), M \in \mathbb{R}$ and $\mu \in \mathbb{C}$. If $p(\mu) \neq 0$, then the function

$$u_P(x) = \frac{Me^{\mu x}}{p(\mu)}$$

satisfies $Lu_P = Me^{\mu x}$.

Example A particular solution of $u'' + 4u' - 3i = 3e^{2x}$ is $u_P = e^{2x}/3$.

Product of Polynomial and Exponential Let $L = p(D)$ and assume that $p(\mu) \neq 0$. For any integer $r \geq 0$, there exists a unique polynomial v of degree r such that $u_P = v(x)e^{\mu x}$ satisfies $Lu_P = x^r e^{\mu x}$.

1.5.2 Annihilator Method

In the previous cases we proposed a solution $u = u_P$ and showed that it satisfied $Lu = f$. The following is a method to derive a particular solution given $Lu = f$. If $f(x)$ is differentiable at least n times and

$$[a_n D^n + a_{n-1} D^{n-1} + \cdots a_1 D^1 + a_0] f(x) = 0$$

then $[a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D^1 + a_0]$ **annihilates** f.

Example D^n annhihilates x^{m-1} for $m \leq n$.
$(D-\alpha)^n$ annihilates $x^{m-1}e^{\alpha x}$ for $m \leq n$.

Annhilator Method: Simple Example Given $Lu = f$ we can apply the appropriate annhiliator to both sides and solving the resutling homogenous DE.
Let $Lu = u'' - u'$ and suppose we want a solution such that $Lu = x^2$. Annihilating both sides we have

$$D^3(u'' - u') = u^{(5)} - u^{(4)} = 0.$$

Setting $w = u^{(4)}$, clearly $w = Ce^x$ is the general solution. Integrating four times yields

$$u = Ce^x + Ex^3 + Fx^2 + Gx + H.$$

Clearly $u_h = Ae^x + H$ and the form of the particular solution is $u_P = x(Ex^2 + Fx + G)$. Substituting we find $E = -1/3, F = -1$ and $G = -2$.

1.5.3 Judicious Guessing Method Continued

Polynomial Solutions: The Reamining Case Let $L = p(D)$ and assume $p(0) = p'(0) = \cdots = p^{(k-1)}(0) = 0$ but $p^{(k)}(0) \neq 0$ where $1 \leq k \leq m-1$. For any integer $r \geq 0$, there exists a unique polynomial v of degree $r$such that $u_P(x) = x^K v(x)$ satisfies $Lu_P = x^r$.

Exponential Times Polynomial: Remaining Case Let $L = p(D)$ and assume $p(\mu) = p'(\mu) = \cdots = p^{(k-1)}(\mu) = 0$. But $p^{(k)}(\mu) \neq 0$, where $1 \leq k \leq m-1$. For any integer $r \geq 0$, there exists a unique polynomial v of degree r such that $u_P(x) = x^k v(x) e^{\mu x}$ satisfies $Lu_P = x^r e^{\mu x}$.

1.5.4 Variation of Parameters

Example Find the general solution to $u'' - 4u' + 4u = (x+1)\exp 2x$.
Note first that the general solution, u_h, to $u'' - 4u' + 4u = 0$ is

$$u(x) = c_1 e^{2x} + c_2 x e^{2x}$$

since the characteristic equation is $0 = r^2 - 4r + 4 = (r-2)^2$. Then

$$W(x) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = e^{4x}.$$

So by the method of variation of parameters:

$$v_1'(x) = e^{-4x} \cdot -xe^{2x}(x+1)e^{2x} \text{ and } v_2'(x) = e^{-4x} \cdot e^{2x}(x+1)e^{2x}.$$

In other words,

$$v_1'(x) = -x^2 - x \text{ and } v_2'(x) = x + 1.$$

Therefore $u(x) = c_1 e^{2x} + c_2 x e^{2x} - (\frac{1}{3}x^3 + \frac{1}{2}x^2)e^{2x} + (\frac{1}{2}x^2 + x)xe^{2x}$.

1.6 Solution via Power Series

General Case Consider a general second-order, linear, homogenous ODE

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u = 0.$$

Equivalently,

$$u'' + p(x)u' + q(x)u = 0,$$

where

$$p(x) = \frac{a_1(x)}{a_2(x)} \text{ and } q(x) = \frac{a_0(x)}{a_2(x)}.$$

Assume that a_j is **analytic** at 0 for $0 \leq j \leq 2$. Then p and q are analytic at 0, that is, they admit power series expansions

$$p(z) = \sum_{k=0}^{\infty} p_k z^k \text{ and } q(z) = \sum_{k=0}^{\infty} q_k z^k \text{ for } |z| < \rho,$$

for some $\rho > 0$.

Convergence Theorem If the coefficients $p(z)$ and $q(z)$ are analytic for $|z| < \rho$, then the formal power series for the solution $u(z)$, constructed above, is also analytic for $|z| < \rho$.

Power Series at Zero Consider

$$Lu = (1 - x^2)u'' - 5xu' - 4u = 0, \quad u(0) = 1, \quad u'(0) = 2.$$

In this case,

$$p(z) = \frac{-5z}{1 - z^2} = -5 \sum_{k=0}^{\infty} z^{2k+1} \text{ and } q(z) = \frac{-4}{1 - z^2} = -4 \sum_{k=0}^{\infty} z^{2k}$$

are analytic for $|z| < 1$, so the theorem guarantees that $u(z)$, given by the formal power series, is also analytic for $|z| < 1$.

Expansion about a Point other than Zero Suppose we want a power series expansion about a point $c \neq 0$, for instance because the initial conditions are given at $x = c$.
A simple change of the independent variable allows us to write

$$u = \sum_{k=0}^{\infty} A_k (z - c)^k = \sum_{k=0}^{\infty} A_k Z^k \text{ where } Z = z - c.$$

Since $du/dx = du/dZ$ and $d^2u/dz^2 = d^2u/dZ^2$, we obtain the translated equation

$$\frac{d^2u}{dZ^2} + p(Z + c)\frac{du}{dZ} + q(Z + c)u = 0.$$

Now compute that A_k using the series expansions of $p(Z + c)$ and $q(Z + c)$ in powers of Z.

1.7 Singular ODEs

In general, we do not want L to be singular on an interval for which we wish to solve $Lu = f$. However, some important applications lead to singular ODEs so we now address this case.

A second-order **Euler-Cauchy ODE** has the form

$$Lu = ax^2u'' + bxu' + cu = f(x),$$

where a, b and c are constants with $a \neq 0$. This ODE is singular at $x = 0$.
Noticing that

$$Lx^r = [ar(r-1) + br + c]x^r,$$

we see that $u = x^r$ is a solution of the homogenous equation ($f = 0$) iff

$$ar(r-1) + br + c = 0.$$

Factorisation Suppose $ar(r-1) + br + c = a(r - r_1)(r - r_2)$. If $r_1 \neq r_2$ then the general solution of the homogenous equation $Lu = 0$ is

$$u(x) = C_1x^{r_1} + C_2x^{r_2}, \quad x > 0.$$

Lemma If $r_1 = r_2$ then the general solution of the homogenous Euler-Cauchy equation $Lu = 0$ is

$$u(x) = C_1x^{r_1} + C_2x^{r_1}\ln x, \quad x > 0.$$

Euler-Cauchy Equations with Nonreal Indicial Roots Suppose that $r_{1,2} = \alpha \pm \beta i$ are the roots of the indicial equation

$$ar(r-1) + br + c = 0$$

associated to the Euler-Cauchy equation

$$at^2u'' + btu' + cu = 0.$$

Then the real-valued solutions can be derived as follows. First note that

$$t^{\alpha+\beta i} = t^\alpha t^{\beta i}$$

is a solution. Then notice that

$$t^{\beta i} = e^{\ln t^{\beta i}} = e^{i \ln t^\beta} = \cos\big(\ln\big(t^\beta\big)\big) + i \sin\big(\ln\big(t^\beta\big)\big).$$

So,

$$t^\alpha t^{\beta i} = t^\alpha e^{\ln t^{\beta i}} = t^\alpha e^{i \ln t^\beta} = t^\alpha \left(\cos\big(\ln\big(t^\beta\big)\big) + i \sin\big(\ln\big(t^\beta\big)\big)\right)$$

is a solution. Finally, since each of the real part an the imaginary part is .separately a (linear independent) solution, we see that the general solution in this case is (for $t > 0$)

$$u(t) = t^\alpha \left(c_1 \cos\big(\ln\big(t^\beta\big)\big) + i \sin\big(\ln\big(t^\beta\big)\big)\right).$$

Example Consider $t^2u'' - tu' + 5u = 0$. Then the indicial equation is

$$r(r-1) - r + 5 = 0 \implies r = 1 \pm 2i.$$

So the general solution is,

$$u(t) = t(c_1 \cos \ln t^2 + c_2 \sin \ln t^2).$$

A number of important applications lead to ODEs that can be written in the **Frobenious normal form**

$$z^2u'' + zP(z)u' + Q(z)u = 0,$$

where $P(z)$ and $Q(z)$ are analytic at $z = 0$:

$$P(z) = \sum_{k=0}^{\infty} P_k z^k \text{ and } Q(z) = \sum_{k=0}^{\infty} Q_k z^k, \quad |z| < \rho.$$

Now consider $z^2u'' + zP(z)u' + Q(z)u = 0$. Formal manipulations show that $Lu(z)$ equals

$$I(r)A_0z^r + \sum_{k=1}^{\infty} \left(I(k+r)A_k + \sum_{j=0}^{k-1} \left[(j+r)P_{k-j} + Q_{k-j}\right] A_j \right) z^{k+r},$$

where $I(r)$ is the indicial polynomial $I(r) := r(r-1)P_0r + Q_0$, so we define $A_0(r) = 1$ and

$$A_k(r) = \frac{-1}{I(k+r)} \sum_{j=0}^{k-1} [(j+r)P_{k-j} + Q_{k-j}]A_j(r), \quad k \geq 1,$$

provided $I(k+r) \neq 0$ for all $k \geq 1$.

1.8 Bessel and Legendre Equations

1.8.1 Bessel Equations and Functions

The **Bessel equation with parameter** ν is

$$z^2u'' + zu' + (z^2 - \nu^2)u = 0.$$

This ODE is in Frobenius normal form, with indicial polynomial $I(r) = (r+\nu)(r-\nu)$, and we seek a series solution

$$u(z) = \sum_{k=0}^{\infty} A_k z^{k+r}.$$

We assume $\mathrm{Re}\,\nu \geq 0$, so $r_1 = \nu$ and $r_2 = -\nu$.
With the normalisation

$$A_0 = \frac{1}{2^\nu \Gamma(1+\nu)}$$

the series solution is called the **Bessel function of order** ν and is denoted

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+\nu)} \left[1 - \frac{(z/2)^2}{1+\nu} + \frac{(z/2)^4}{2!(1+\nu)(2+v)} - \cdots \right].$$

From the functional equation $\Gamma(1+z) = z\Gamma(z)$ we see that

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(1+v)} - \frac{(z/2)^{\nu+2}}{\Gamma(2+\nu)} + \frac{(z/2)^{\nu+4}}{2!\Gamma(3+v)} - \frac{(z/2)^{\nu+6}}{3!\Gamma(4+\nu)} + \cdots$$

and so

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\Gamma(k+1+\nu)}.$$

1.8.2 Legendre Equation

The **Legendre equation** with parameter nu is

$$(1-z^2)u'' - 2zu' + \nu(\nu+1)u = 0.$$

This ODE is not singular at $z=0$ so the solution has an ordinary Taylor series expansion

$$u = \sum_{k=0}^{\infty} A_k z^k.$$

The A_k must satisfy

$$(k+1)(k+2)A_{k+2} - [k(k+1) - \nu(\nu+1)A_k] = 0$$

for $k \geq 0$, and since

$$k(k+1) - \nu(\nu+1) = (k-\nu)(k+\nu+1),$$

the recurrence relation is

$$A_{k+1} = \frac{(k-\nu)(k+\nu+1)}{(k+1)(k+2)} A_k \text{ for } k \geq 0.$$

We have

$$u(z) = A_0 u_0(z) + A_1 u_1(z)$$

where

$$u_0(z) = 1 - \frac{\nu(\nu+1)}{2!} z^2 + \frac{(\nu-2)\nu(\nu+1)(\nu+3)}{4!} z^4 - \cdots$$

and

$$u_1(z) = z - \frac{(\nu-1)(\nu+2)}{3!} z^3 + \frac{(\nu-3)(\nu-1)(\nu+2)(\nu+4)}{5!} z^5 - \cdots .$$

Suppose now that $\nu = n$ is a non-negative integer. If n is even the series for $u_0(z)$ terminates, whereas if n is odd then the series for $u_1(z)$ terminates.

The terminating solution is called the **Legendre polynomial** of degree n and is denoted by $P_n(z)$ with the normalization

$$P_n(1) = 1.$$

Lengdre Polynomials The first few Legendre polynomials are

$$\begin{aligned} P_0(z) &= 1, & P_3(z) &= \frac{1}{2}(5z^3 - 3z), \\ P_1(z) &= z, & P_4(z) &= \frac{1}{8}(35z^4 - 30z^2 + 3), \\ P_2(z) &= \frac{1}{2}(3z^2 - 1), & P_5(z) &= \frac{1}{8}(63z^5 - 70z^3 + 15z). \end{aligned}$$

Notice that P_n is an even or odd function according to whether n is even or odd.

Chapter 2

Dynamical Systems

2.1 Terminology

We begin with some examples of how systems of differential equations arise in applications, and see how all such problems can be formulas as a **first-order** system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}).$$

Such a formulation leads to a natural geometric interpretation of a solution.

Lotka-Volterra Equations Simplified ecology with two species:

$$\begin{aligned} F(t) &= \text{ number of foxes at time t}, \\ R(t) &= \text{ number of rabbits at time t}. \end{aligned}$$

Assume populations large enough at F and R can be treated as smoothly varying in time.
In the 1920s, Alfred Lotka and Vito Volterra independently proposed the predator-prey model

$$\begin{aligned} \frac{dF}{dt} &= -aF + \alpha FR, \quad F(0) = F_0, \\ \frac{dR}{dt} &= bR - \beta FR, \qquad R(0) = R_0. \end{aligned}$$

Here a, α, b and β are non-negative constants.

Any first-order system for N ODEs in the form

$$\begin{aligned} \frac{dx}{dy} &= F_1(x, y, \ldots, x_N), \quad x(0) = x_{10}, \\ \frac{dy}{dt} &= F_2(x, y, \ldots, x_N), \quad y(0) = x_{20}, \\ &\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ \frac{dx_N}{dt} &= F_N(x, y, \ldots, x_N), \quad x_N(0) = x_{N0}, \end{aligned}$$

can be written in vector notation as

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \mathbf{x}(0) = \mathbf{x}_0.$$

The system of ODEs is determined by the **vector field** $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^N$.
A system of ODEs of the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

is said to be **autonomous**.
In a **non-autonomous** system, $\mathbf{F}$ will depend explicitly on t:

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t).$$

It can be shown that it is sufficient (in principle) to develop theory for the autonomous case as a non-autonomous system can be converted into an autonomous system.

Second-order ODE Consider an initial-value problem for a general (possibly non-autonomous) second-order ODE

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right), \text{ with } x = x_0 \text{ and } \frac{dx}{dt} = y_0 \text{ at } t = 0.$$

If $x = x(t)$ is a solution, and if we let $y = dx/dt$, then

$$\frac{dy}{dt} = \frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}, t\right) = f(x, y, t),$$

that is, (x, y) is a solution of the first-order system

$$\begin{aligned} \frac{dx}{dt} &= y, & x(0) &= x_0, \\ \frac{dy}{dt} &= f(x, y, t) & y(0) &= y_0. \end{aligned}$$

2.2 Existence and Uniqueness

The most fundamental question about a dynamical system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t)$$

is

> For a given initial value $\mathbf{x}_0$, does a solution $\mathbf{x}(t)$ satisfying $\mathbf{x}(0) = \mathbf{x}_0$ exist, and if so is this solution unique?

Answer is **yes**, whenever the vector field $\mathbf{F}$ is **Lipschitz**.

The number L is a **Lipschitz constant** for a function $f : [a, b] \to \mathbb{R}$ if

$$|f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y \in [a, b].$$

Example Consider $f(x) = 2x^2 - x + 1$ for $0 \leq x \leq 1$. Since

$$\begin{aligned} f(x) - f(y) = 2(x^2 - y^2) - (x - y) &= 2(x + y)(x - y) - (x - y) \\ (2x + 2y - 1)(x - y) \end{aligned}$$

we have $|f(x)-f(y)| = |2x+2y-1||x-y|$ so a Lipschitz constant is

$$L = \max_{x,y\in[0,1]} |2x+2y-1| = 3.$$

We say that the function $f : [a, b] \to \mathbb{R}$ is Lipschitz if a Lipschitz constant for f exists.

Lipschitz Continuity If f is Lipshictz then f is (uniformly) continuous.

Continous does not imply Lipschitz Consider the (uniformly) continous function

$$f(x) = 3 + \sqrt{x} \text{ for } 0 \le x \le 4.$$

In this case, if $x, y \in (0, 4]$ then

$$\begin{aligned} f(x) - f(y) = \sqrt{x} - \sqrt{y} &= \left(\sqrt{x} - \sqrt{y} \times \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right) \\ &= \frac{x-y}{\sqrt{x}+\sqrt{y}} \end{aligned}$$

so if a Lipschitz constant L exists then

$$L \ge \frac{|f(x)-f(y)|}{|x-y|} = \frac{1}{\sqrt{x}+\sqrt{y}}$$

for arbitrarily small x and y, a contradiction.

A function $f : I \to \mathbb{R}$ is C^k if $f, f', f'', \dots, f^{(k)}$ all exist and are continuos on the interval I.

Theorem For any closed and bounded interval $I = [a, b]$, if f is C^1 on I then $L = \max_{x\in I} |f'(X)|$ is a Lipschitz constant for f on I.

A vector field $\mathbf{F} : S \in \mathbb{R}^N$ is Lipschitz on $S \subseteq \mathbb{R}^N$ if

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{y})\| \le L\|\mathbf{x}-\mathbf{y}\| \quad \text{for all} \mathbf{x}, \mathbf{y} \in S$$

Here,

$$\|\mathbf{x}\| = \left(\sum_{j=1}^{N} x_j^2\right)^{\frac{1}{2}}$$

denotes the **Euclidean norm** of the vector $\mathbf{x} \in \mathbb{R}^N$.
We say that $\mathbf{F}(\mathbf{x}, t)$ is **Lipschitz in x** if

$$\|\mathbf{F}(\mathbf{x},t) - \mathbf{F}(\mathbf{y},t)\| \le L\|\mathbf{x}-\mathbf{y}\|.$$

Local Existence and Uniquness Let $\mathbf{x}_0 \in \mathbb{R}^N$, fix $r > 0$ and $\tau > 0$, and put

$$S = \{(\mathbf{x}, t) \in \mathbb{R}^N \times \mathbb{R} : \|\mathbf{x}-\mathbf{x}_0\| \le r \text{ and } |t| \le \tau\}.$$

If $\mathbf{F}(\mathbf{x}, t)$ is Lipschitz in $\mathbf{x}$ for $\mathbf{x}, t \in S$, and if

$$\|\mathbf{F}(\mathbf{x}, t)\| \leq M \quad \text{for } (\mathbf{x}, t) \in S,$$

then there exists a unique C^1 function $bfx(t)$ satisfying

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t) \quad \text{for } |t| \leq \min\{r/M, \tau\}, \text{ with } \mathbf{x}(0) = \mathbf{x}_0.$$

2.3 Linear Dynamical Systems

Linear differential equations are generally much easier to solve than nonlinear ones. Fortunately, linear DEs suffice for describing many important applications.
We say that the $N \times N$, first order system of ODEs

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, t)$$

is **linear** if the RHS has the form

$$\mathbf{F}(\mathbf{x}, t) = A(t)\mathbf{x} + \mathbf{b}(t)$$

for some $N \times N$ matrix-valued function $A(t) = [a_{ij}(t)]$ and a vector-valued function $\mathbf{b} = [b_i(t)]$.

The system is autonomous precisely when A and $\mathbf{b}$ are constant.

Global Existence and Uniqueness If the elements of $A(t)$ and components of $\mathbf{b}$ are continuous for $0 \leq t \leq T$, then the linear initial-value problem

$$\frac{d\mathbf{x}}{dt} = A(t)\mathbf{x} + \mathbf{b}(t) \quad \text{for } 0 \leq t \leq T, \quad \text{with } \mathbf{x}(0) = \mathbf{x}_0,$$

has a unique solution $\mathbf{x}(t)$ for $0 \leq t \leq T$.

We now investigate the special case when A is constant and $\mathbf{b}(t) = \mathbf{0}$:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}$$

General Solution via Eigensystem If $\mathbf{v}$ is a constant vector and $A\mathbf{v} = \lambda\mathbf{v}$, we define $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$. Then

$$\frac{d\mathbf{x}}{dt} = \lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}(\lambda\mathbf{v}) = e^{\lambda t}(A\mathbf{v}) = A(e^{\lambda t}\mathbf{v} = A\mathbf{x})$$

that is, $\mathbf{x}$ is a solution of $dvx/dt = A\mathbf{x}$. If $A\mathbf{v}_j = \lambda_j\mathbf{v}_j$ for $1 \leq j \leq N$, then the linear combination

$$\mathbf{x}(t) = \sum_{j=1}^{N} c_j e^{\lambda_j} t\mathbf{v}_j$$

is also a solution because the ODE is linear and homogenous. Provided the $\mathbf{v}_j$ are linearly independent, then the above equation is a **general solution** because given any $x_0 \in \mathbb{R}^N$ there exist unique c_j such that

$$\mathbf{x}(0) = \sum_{j=1}^{N} c_j\mathbf{v}_j = \mathbf{x}_0.$$

Example Consider

$$\frac{dx}{dt} = -5x + 2y, \quad x(0) = 5,$$
$$\frac{dy}{dt} = -6x + 3y \quad\quad y(0) = 7.$$

Note that the initial value problem can be written in the vector form

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t), \mathbf{x}(0) = \mathbf{x}_0,$$

$$\mathbf{A} = \begin{bmatrix} -5 & 2 \\ -6 & 3 \end{bmatrix} \quad \text{and } \mathbf{x}_0 := \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

Solving the system, using the eigenpair approach, we would need to find the eigenvectors and eigenvalues.
Characteristic equation is

$$0 = |\mathbf{A} - \lambda \mathbf{I}| = (-5-\lambda)(3-\lambda) + 12 \implies \lambda_1 := -3 \text{ and } \lambda_2 = 1.$$

Next we find the associated eigenvectors.

$$\lambda_1 = -3 : (\mathbf{A} + 3\mathbf{I})\mathbf{v} = \mathbf{0} \implies \mathbf{v}_1 := \begin{bmatrix} 1 \\ 1 \end{bmatrix} \lambda_2 \qquad = 1 : (\mathbf{A} + \mathbf{I})\mathbf{v} = \mathbf{0} \implies \mathbf{v}_1 := \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

This means that a general solution of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is

$$\mathbf{x}(t) = c_1 e^{-3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Applying the initial value we can see that the unique solution is $x(t) = 4e^{-3t} + e^t$ and $y(t) = 4e^{-3t} + 3e^t$.

A square matrix $A \in \mathbb{C}^{N \times N}$ is **diagonalisable** if there exists a non-singular matrix $Q \in \mathbb{C}^{N \times N}$ such that $Q^{-1}AQ$ is diagonal.

Theorem A square matrix $A \in \mathbb{C}^{N \times N}$ is diagonalisable if and only if there exists a basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N\}$ for $\mathbb{C}^N$ consisting of eigenvectors of A. Indeed if,

$$A\mathbf{v}_j = \lambda_j \mathbf{v}_j \text{ for } j = 1, 2, \dots, N,$$

and we put $Q = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots \mathbf{v}_N \end{bmatrix}$ then $Q^{-1}AQ = A$ where

$$A = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{bmatrix}$$

Consider a diagonalisable matrix A. Since $Q^{-1}AQ = \Lambda$, it follows that A has an eigenvalue decomposition

$$A = Q\Lambda Q^{-1}.$$

In general, we see by induction on k that

$$A^k = Q\Lambda^k Q^{-1} \text{ for } k = 0, 1, 2, \dots$$

Example

$$A = \begin{bmatrix} -5 & 2 \\ -6 & 3 \end{bmatrix}$$

then

$$\Lambda = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad Q^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

so

$$A^k = Q\Lambda^k Q^{-1} = \frac{1}{2}\begin{bmatrix} (-1)^k \times 3^{k+1} - 1 & (-1)^{k+1} \times 3^k + 1 \\ (-1)^k \times 3^{k+1} - 3 & (-1)^{k+1} \times 3^k + 3 \end{bmatrix}.$$

For any polynomial

$$p(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_m z^m$$

and any square matrix A, we define

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_m A^m.$$

When A is diagonalisable, $A^k = Q\Lambda^k Q^{-1}$ so

$$\begin{aligned} p(A) &= c_0 QIQ^{-1} + c_1 Q\Lambda Q^{-1} + \cdots + c_m Q\Lambda^m Q^{-1} \\ &\vdots \\ &= Qp(\Lambda)Q^{-1} \end{aligned}$$

Lemma For any polynomial p and any diagonal matrix Λ,

$$p(A) = \begin{bmatrix} p(\lambda_1) & & \\ & \ddots & \\ & & p(\lambda_N) \end{bmatrix}$$

Theorem If two polynomials p and q are equal on the specturm of a diagonlisable matrix A, that is, if

$$p(\lambda_j) = q(\lambda_j) \text{ for } j = 1, 2, \ldots, N,$$

then $p(A) = q(A)$.

Example Recall that

$$A = \begin{bmatrix} -5 & 2 \\ -6 & 3 \end{bmatrix}$$

has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 1$. Let

$$p(z) = z^2 - 4 \quad \text{and} \quad q(z) = -2z - 1,$$

and observe

$$p(-3) = 5 = q(-3) \text{ and } p(1) = -3 = q(1).$$

We find

$$p(A) = A^2 - 4I = \begin{bmatrix} 9 & -4 \\ 12 & -7 \end{bmatrix} = -2A - I = q(A).$$

Exponential of a Diagonalisable Matrix If $A = Q\Lambda Q^{-1}$ is diagonalisable, then

$$e^A = Qe^\Lambda Q^{-1} \text{ and } e^\Lambda = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_N} \end{bmatrix}$$

Given a forced linear system of the form $\mathbf{x}' - \mathbf{A}\mathbf{x}(t) = \mathbf{f}(t)$. We can use the **variation of constants formula** to solve the vector equation.

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 + e^{\mathbf{A}t}\int_0^t e^{-\mathbf{A}s}\mathbf{f}(s)\,ds.$$

Fundamental Matrix A fundamental matrix Φ for the linear homogenous vector equation

$$\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$$

satisfies the following two properties.

1. The columns of $\mathbf{X}$ are linearly independent vector functions so that, in particular, $|\mathbf{X}(t)| \neq 0$; and
2. Φ solves the matrix equation $\mathbf{X}'(t) = \mathbf{A}\mathbf{X}(t)$.

Theorem Suppose that $\mathbf{\Phi}$ is a fundamental matrix for the vector equations

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

Then every solution of this equation has the form

$$\mathbf{\Phi}\mathbf{c}$$

for some constant vector $\mathbf{c}$.

Nilpotent Matrix A matrix is nilpotent if there exists a positive integer k such that $\mathbf{A}^k = \mathbf{O}$, where $\mathbf{O}$ denotes the zero matrix.

If $\mathbf{A}$ is nilpotent then we can easily find $e^{\mathbf{A}t}$. In particular,

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$$

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} + \frac{1}{2}t^2\mathbf{A}^2 + \dots$$

Example

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{O}.$$

Therefore $\mathbf{A}$ is nilpotent and, in particular,

$$e^{\mathbf{A}t} = \mathbf{I} + t\mathbf{A} = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$$

2.4 Stability

In many applications we are interested to know how the solution $\mathbf{x}(t)$ behaves as $t \to \infty$, and might not care much about the precise details of the transient behaviour for finite t. We say that $\mathbf{a} \in \mathbb{R}^N$ is an equilibrium point for the dynamical system $d\mathbf{x}/dt = \mathbf{F}(\mathbf{x})$ if

$$\mathbf{F}(\mathbf{a}) = \mathbf{0}.$$

Thus the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \text{for all } t, \text{ with } \mathbf{x}(0) = \mathbf{a}$$

is just the constant function $\mathbf{x}(t) = \mathbf{a}$.
An equilibrium point $\mathbf{a}$ is **stable** if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $\|\mathbf{a}_0 - \mathbf{a}\| < \delta$ the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \text{for } t > 0, \text{ with } \mathbf{x}(0) = \mathbf{x}_0$$

satisfies

$$\|\mathbf{x}(t) - \mathbf{a}\| < \epsilon \text{ for all } t > 0.$$

Let D be an open subset of $\mathbb{R}^N$ that contains an equilibrium point $\mathbf{a}$. We say that $\mathbf{a}$ is **asymptotically stable** in D if $\mathbf{a}$ is stable and, whenever $\mathbf{a}_0 \in D$, the solution of

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) \quad \text{for } t > 0, \text{ with } \mathbf{x}(0) = \mathbf{x}_0$$

satisfies

$$\mathbf{a}(t) \to \mathbf{a} \text{ as } t \to \infty.$$

In this case D is called a **domain of attraction** for $\mathbf{a}$.

Criteria for Stability Let A be a diagonlisable matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. The equilibrium point $\mathbf{a} = -A^{-1}\mathbf{b}$ is of

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b} \text{ with } \mathbf{x}(0) = \mathbf{x}_0 \text{ and } \det(A) \neq 0.$$

1. **stable** if and only if $\operatorname{Re}\lambda_j \leq 0$ for all j
2. **asymptotically stable** if and only if $\operatorname{Re}\lambda_j < 0$ for all j.

In the second case, the domain of attraction is the whole of $\mathbb{R}^N$.

2.5 Classification of 2D Linear Systems with $\det A \neq 0$

The equilibrium point $\mathbf{a} = 0$ may be asymptotically stable, stable or unstable but may also have various other properties.

2.5.1 Case 1: Real Eigenvalues and Linearly Independent Eigenvectors

Suppose you have real eigenvalues λ_1 and λ_2 and two linearly independent eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
General solution:

$$\mathbf{x} = c_1 e^{\lambda_1 t}\mathbf{v}_1 + c_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Canonical form:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

Stable Node Example ($\lambda_2 < \lambda_1 < 0$)

$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = -2y, A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = -1, \lambda_2 = -2, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The general solution is

$$\mathbf{x} = c_1 e^{-t}\mathbf{v}_1 + c_2 e^{-2t}\mathbf{v}_2 = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-2t} \end{pmatrix}.$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(0)e^{-t} \\ y(0)e^{-2t} \end{pmatrix}.$$

Unstable Node Example ($0 < \lambda_1 < \lambda_2$)

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = 2y, A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = 1, \lambda_2 = 2, \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(0)e^{t} \\ y(0)e^{2t} \end{pmatrix}.$$

All trajectories (except $\mathbf{x}(t) = \mathbf{0}$) are repelled from equilibrium point which is unstable.

(Un)stable Stars ($\lambda_1 = \lambda_2 \neq 0$)

$$\frac{dx}{dt} = \lambda_1 x, \quad \frac{dy}{dt} = \lambda_1 y, A = \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

All vectors are eigenvectors.
The general solution is

$$\mathbf{x} = e^{\lambda_1 t}\mathbf{v}.$$

Solution of the initial value problem:

$$\mathbf{x}(t) = e^{\lambda_1 t}\mathbf{x}(0).$$

All orbits (except $vx(t) = \mathbf{0}$) are oriented half-lines which are either attracted $\lambda_1 < 0$ or repelled ($\lambda_1 > 0$) by the equilibrium point.

Saddle Node Example (unstable: $\lambda_2 < 0 < \lambda_1$)

$$\frac{dx}{dt} = x + 2y, \quad \frac{dy}{dt} = 3x + 2y, A = \lambda_1 \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

Eigenvalues and eigenvectors:

$$\lambda_1 = -1, \lambda_2 = 4, \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Here the first solution is repelling and the section is attracting, so the solution is unstable.

Nonreal eigenstuff for A

$$\begin{aligned} e^{\lambda_1 t}\mathbf{v}_1 &= e^{(\alpha+\beta i)t}(\mathbf{p} + i\mathbf{q}) \\ &= e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))(\mathbf{p} + i\mathbf{q}) \\ &= \underbrace{e^{\alpha t}(\cos(\beta t)\mathbf{p} - \sin(\beta t)\mathbf{q})}_{:=\mathbf{x}_{\text{Re}}(t)} + i\,\underbrace{e^{\alpha t}(\sin(\beta t)\mathbf{p} + \cos(\beta t)\mathbf{q})}_{:=\mathbf{x}_{\text{Im}}(t)} \end{aligned}$$

So, a basis for the solution space is then

$$\mathcal{B} := \{\mathbf{x}_{\text{Re}}, \mathbf{x}_{\text{Im}}\}.$$

The general solution is,

$$\mathbf{x}(t) := c_1\mathbf{x}_{\text{Re}}(t) + c_2\mathbf{x}_{\text{Im}}(t)$$

for arbitrary constants $c_1, c_2 \in \mathbb{R}$.

2.5.2 Case 2: Complex Conjugate Eigenvalues

Suppose you have complex conjugate eigenvalues $\lambda_1 = \bar{\lambda_2} \notin \mathbb{R}$.
General solution:

$$\mathbf{x} = c_1 \operatorname{Re}(e^{\lambda_1 t}\mathbf{v}_1) + c_2 \operatorname{Im}(e^{\lambda_1 t}\mathbf{v}_1).$$

Canonical form:

$$A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \lambda_1 = \alpha + i\beta.$$

Interpretation

$$\mathbf{x}(t) = e^{\alpha t}R(t)\mathbf{x}(0), \quad R(t) = \begin{pmatrix} \cos\beta t & \sin\beta t \\ -\sin\beta t & \cos\beta t \end{pmatrix}$$

Thus, the initial vector $\mathbf{x}(0)$ is rotated by the rotation matrix $R(t)$ and scaled by the factor $e^{\alpha t}$.

Centre Example (stable: $\text{Re}(\lambda_1) = 0$**)**

$$\frac{dx}{dt} = -2y, \quad \frac{dy}{dt} = 2x, A = \lambda_1 \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$$

Eigenvalues:

$$\lambda_1 = \bar{\lambda_2} = -2i$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} y(0) \\ y(0) \end{pmatrix}.$$

The solution constitutes orbits which are oriented circles. These are stable (but not asymptotically stable).

Stable Foci Example **(**$\text{Re}(\lambda_1) < 0$**)**

$$\frac{dx}{dt} = -x - 2y, \quad \frac{dy}{dt} = 2x - y, A = \lambda_1 \begin{pmatrix} -1 & -2 \\ 2 & -1 \end{pmatrix}$$

Eigenvalues:

$$\lambda_1 = \bar{\lambda_2} = -1 - 2i$$

Solution of the initial value problem:

$$\begin{pmatrix} x(y) \\ y(t) \end{pmatrix} = e^{-t} \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} \to \mathbf{0} \text{ as } t \to \infty.$$

Orbits are oriented spirals which are attracted to the asymptotically stable equilibrium point.

2.6 Final Remarks on Nonlinear DEs

A function $G : \mathbb{R}^N \to \mathbb{R}$ is a **first integral** (or constant of the motion) for the system of ODEs

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

if $G(\mathbf{x}(t))$ is constant for every solution $\mathbf{x}(t)$.

Simple Example The function $G(x, y) = x^2 + y^2$ is a first integral of the linear system of ODEs

$$\frac{dx}{dt} = -y, \quad \frac{dy}{dt} = x.$$

In fact, putting

$$\mathbf{F}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

we have

$$\boldsymbol{\nabla} \cdot \mathbf{F} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \cdot \begin{bmatrix} -y \\ x \end{bmatrix} = (2x)(-y) + (2y)(x) = 0,$$

or equivalently,

$$\frac{dG}{dt} = \frac{\partial G}{\partial x}\frac{dx}{dt} + \frac{\partial G}{\partial y}\frac{dy}{dt} = (2x)(-y) + (2y)(x) = 0.$$

Cayley-Hamilton Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Then $\mathbf{A}$ satisfies its characteristic equation.

Putzer's Algorithm Let $\{\lambda_j\}_{j=1}^n$ be the collection of n not necessarily distinct eigenvalues of a given matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$. Then

$$e^{\mathbf{A}t} = \sum_{k=0}^{n-1} p_{k+1}(t)\mathbf{M}_k,$$

where

$$\mathbf{M}_0 := \mathbf{I} \text{ and } \mathbf{M}_k := \Pi_{j=1}^k(\mathbf{A} - \lambda_j\mathbf{I}), 1 \le k \le n,$$

and the vector-valued function $\mathbf{p}(t) := (p_1(t), \ldots, p_n(t))$ satisfies the vectorial equation

$$\mathbf{p}'(t) = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0\cdots & 0 & \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} \mathbf{p}(t), \quad \mathbf{p}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

So in the case in which $n = 2$, i.e., a two-dimensional vector equation, Putzer's algorithm reduces to

$$e^{\mathbf{A}t} = p_1(t)\mathbf{I} + p_2(t)(\mathbf{A} - \lambda_1\mathbf{I}),$$

where

$$\begin{bmatrix} p_1'(t) \\ p_2'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 1 & \lambda_2 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}, \quad \mathbf{p}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Similarly, in the case in which $n = 3$, i.e., a three-dimensional vector equation, Putzer's algorithm reduces to

$$e^{\mathbf{A}t} = p_1(t)\mathbf{I} + p_2(t)(\mathbf{A} - \lambda_1\mathbf{I}) + p_3(t)(\mathbf{A} - \lambda_1\mathbf{I})(\mathbf{A} - \lambda_2\mathbf{I}),$$

where

$$\begin{bmatrix} p_1'(t) \\ p_2'(t) \\ p_3'(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 1 & \lambda_2 & 0 \\ 0 & 1 & \lambda_3 \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{bmatrix}, \quad \mathbf{p}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Chapter 3

Initial-Boundary Value Problems in 1D

We have seen that an initial-value problem for a (nonsingular) linear ODE $Lu = f$ always has a unique solution. However, matters are not so simple for a **boundary-value problem**: a solution might not exist, or if one exists it might not be unique.

3.1 Two-Point Boundary Value Problems

In an mth order **initial-value problem** we specify m initial conditions at the left end of the interval. In an mth order **boundary-value problem**, we again specify m conditions involving the solution and its derivatives, but some apply at the left end and some at the right end.

Boundary Conditions Consider the second-order ODE

$$u'' + u' = 0 \quad \text{for } 0 < x < \pi$$

whose general solution is

$$u(x) = A\cos x + B\sin x.$$

A **unique solution** $u(x) = \sin x$ exists satisfying

$$u'(0) = 1 \text{ and } u(\pi) = 0.$$

No solution exists satisfying

$$u'(0) = 0 \text{ and } u(\pi) = 1.$$

Infinitely many solutions $u(x) = C\sin x$ exists satisfying

$$u'(0) = 0 \text{ and } u(\pi) = 0.$$

We want to solve (Inhomogeneous BVP):

$$Lu = f \quad \text{for } a < x < b, \quad \text{with } B_1 u = \alpha_1 \text{ and } B_2 u = \alpha_2,$$

where

$$Lu = a_2 u'' + a_1 u' + a_0 u$$

is a 2nd-order linear differential operator, and the **boundary operators** have the form

$$B_1 u = b_{11} u'(a) + b_{10} u(a),$$
$$B_2 u = b_{21} u'(b) + b_{20} u(b).$$

Linear Two-Point Bounary Value

$$\begin{aligned} u'' - u &= x - 1 && \text{for } 0 < x < \log 2, \\ u &= 2 && \text{at } x = 0, \\ u' - 2u &= 2\log 2 - 4 && \text{at } x = \log 2. \end{aligned}$$

3.2 Existence and Uniqueness

Since L, B_1 and B_2 are all linear, the solutions of the **homogenous BVP**

$$Lu = 0 \quad \text{for } a < x < b, \quad \text{with } B_1 u = 0 \text{ and } B_2 u = 0,$$

form a vector space: if u_1 and u_2 are solutions of the inhomogeneous BVP then so is $u = c_1 u_! + c_2 u_2$ for any constants c_1 and c_2.

Uniqueness The inhomogenous BVP has **at most** one solution iff the homogenous BVP has only the **trivial solution** $u \equiv 0$.

Exactly One Solution If the homogenous problem has only the trivial solution, then for every choice of f, α_1 and α_2 the inhomogenous problem

$$Lu = f \quad \text{for } a < x < b, \quad \text{with } B_1 u = \alpha_1 \text{ and } B_2 u = \alpha_2,$$

has a unique solution.

3.3 Inner Products and Norms of Functions

If a homogenous initial boundary value problem admits non-trivial solutions, then the inhomogeneous problem might or might not have any solutions, depending on the forcing term and boundary values.

To formulate a condition that guarantees existence we require a short digression that introduces some ideas from functional analysis.

The **inner product** $\langle f, g\rangle$ of a pair of continuous functions $f, g : [a, b] \to \mathbb{R}$ is defined by

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$

The corresponding **norm** of f is defined by

$$\|f\| = \sqrt{\langle f, f\rangle} = \left(\int_a^b [f(x)]^2\right)^{1/2}.$$

We say that f and g are **orthogonal** if $\langle f, g\rangle = 0$.

Inner Product and Norms If

$$[a,b]=[-1,1], \quad f(x)=x, \quad g(x)=\cos\pi x,$$

then

$$\langle f,g\rangle=\int_{-1}^{1} x\cos\pi x\,dx=0, \quad \|f\| = \sqrt{\frac{2}{3}}, \quad \|g\|=1.$$

Thus, $f$and g are orthogonal over the interval $[-1,1]$.

Cauchy-Schwarz Inequality $|\langle f,g\rangle \leq \|f\|\,\|g\|$.

Triangle Inequality $\|f+g\| \leq \|f\| + \|g\|$.

3.4 Self-Adjoint Differential Operators

Define the **formal adjoint** as

$$\begin{aligned} L^*v &= (a_2v)''-(a_1v)'+a_0v \\ &= a_2v''+(2a_2'-a_1)v'+(a_2''-a_1'+a_0)v \end{aligned}$$

and the **bilinear concomitant**

$$P(u,v)=u'(a_2v)-u(a_2v)'+u(a_1v),$$

we have the **Lagrange identity**

$$\langle Lu,v\rangle=\langle u,L^*v\rangle+[P(u,v)]_a^b.$$

Adjoint Operators and Lagrange identity If

$$Lu=3xu''-(\cos x)u'+e^xu$$

then

$$\begin{aligned} L^*v &= (3xv)''+[(\cos x)v]'+e^xv \\ &= 3xv''+(6+\cos x)v'+(e^x-\sin x)v \end{aligned}$$

and

$$\begin{aligned} P(u,v) &= u'(3xv)-u(3xv)'-uv\cos x \\ &= 3x(u'v-uv')-(3+\cos x)uv. \end{aligned}$$

Then $(Lu)v=uL^*v+\frac{d}{dx}P(u,v)$.

The operator L is **formally self-adjoint** if $L^*=L$.

Formally Self-Adjoint Condition A second-order, linear differential operator L is formally self-adjoint iff it can be written in the form

$$Lu = -(pu')' + qu = -pu'' - p'u' + qu,$$

in which case the Lagrange identity takes the form

$$(Lu)v - u(Lv) = -(p(x)(u'v - uv'))',$$

or in other words, the bilinear concomitant it

$$P(u, v) = -p(x)(u'v - uv').$$

Bessel and Legendre are Self-Adjoint Conside the Bessel equation

$$x^2u'' + xu' + (x^2 - \nu^2)u = f(x).$$

Dividing both sides by x gives $Lu = -x^{-1}f(x)$ where

$$Lu = -(xu')' + (\nu^2x^{-1} - x)u.$$

The Legendre equation

$$(1 - x^2)u'' - 2xu' + \nu(\nu + 1)u = f(x)$$

has the form $Lu = -f(x)$ with $Lu = -[(1 - x^2)u']u' - \nu(\nu + 1)u$.

Transforming to Formally Self-Adjoint Form If we can evaluate the integrating factor

$$p(x) = \exp\left(\int \frac{a_1(x)}{a_2(x)}dx\right),$$

then we can transform an ODE of the form $a_2u'' + a_1u' + a_0u = f(x)$ to formally self-adjoint form:

$$\begin{aligned} -pu'' - \frac{pa_1}{a_2}u' - \frac{pa_0}{a_2}u &= \frac{-pf(x)}{a_2}, \\ -(pu'' + p'u') - \frac{pa_0}{a_2}u &= \frac{-pf(x)}{a_2}, \\ -(pu')' + qu &= \tilde{f}(x) \end{aligned}$$

where $q = -pa_0/a_2$ and $\tilde{f} = -pf/a_2$.

Euler-Cauchy ODE Write the Euler-Cauchy ODE $ax^2u'' + bxu' + cu = f(x)$ in formally self-adjoint form. Note that here $a_2(x) = ax^2, a_1(x) = bx$ and $a_0(x) \equiv c$.
Define p by

$$p(x) = \exp\left(\int \frac{bx}{ax^2}dx\right) = \exp\left(\frac{b}{a}\int \frac{1}{x}dx\right) = e^{\frac{b}{a}\ln x} = x^{\frac{b}{a}}.$$

Then recalling that

$$q = -\frac{pa_0}{a_2} \text{ and } \tilde{f} = \frac{pf}{a_2},$$

the formally self-adjoint form is

$$-(x^{\frac{b}{a}}u')' - \frac{c}{ax^2}x^{\frac{b}{a}}u = -\frac{1}{ax^2}x^{\frac{b}{a}}f(x).$$

Self-Adjointness and Boundary Operators Any formally self-adjoint operator $L = -(pu')' + qu$ satisfies the identity

$$\langle Lu, v\rangle - \langle u, Lv\rangle = \sum_{i=1}^{2}(B_i u R_i v - R_i u B_i v),$$

for all v and v where

$$R_1 u = \frac{p(a)u(a)}{b_{11}} \text{ or } R_1 u = -\frac{p(a)u'(a)}{b_{10}}$$

and

$$R_2 u = -\frac{p(b)u(b)}{b_{21}} \text{ or } R2_u = \frac{p(b)u'(b)}{b_{20}}.$$

Necessary Condition for Existence If u is a solution of the inhomogeneous BVP, and if v is a solution of the homogenous problem

$$\begin{aligned} Lv &= 0 && \text{for } a < x < b, \\ B_1 v &= 0 && \text{at } x = a, \\ B_2 v &= 0 && \text{at } x = b. \end{aligned}$$

then on the one hand

$$\langle Lv, v\rangle - \langle u, Lv\rangle = \langle f, v\rangle - \langle u, 0\rangle = \langle f, v\rangle$$

and on the other hand,

$$\langle Lv, v\rangle - \langle u, Lv\rangle = \underbrace{\alpha_1}_{=B_1 u} R_1 v - R_1 u \times \underbrace{0}_{=B_1 v} + \underbrace{\alpha_2}_{=B_2 u} R_2 v - R_2 u \times \underbrace{0}_{=B_2 v}.$$

then the data f, α_1 and α_2 must satisfy

$$\langle f, v\rangle = \alpha_1 R_1 v + \alpha_2 R_2 v.$$

Fredholm Alternative Either the homogenous BVP has only the trivial solution $v \equiv 0$, in which case

> the inhomogeneous BVP has a unique solution u for every choice of f, α_1 and α_2,

OR else the homogenous BVP admits non-trivial solutions, in which case

> the inhomogeneous BVP has a solution u iff f, α_1 and α_2 satify $\langle f, v\rangle = \alpha_1 R_1 v + \alpha_2 R_2 v$ for every solution v of the homogenous BVP.

In the latter case, $u + Cv$ is also a solution of the inhomogeneous BVP for any constant C.

Chapter 4

Generalised Fourier Series

This chapter will cover how if we generalise the concept of Fourier expansions that include the familiar trigonometric Fourier series allows us to solve a range of partial differential equations by separating variables in curvilinear coordinates.

4.1 Separation of Variables for Linear PDEs

As an example of the **separation of variables technique for linear PDEs** consider the one-dimensional heat PDE, which is

$$u_t = c^2 u_{xx},$$

where c is the thermal diffusivity of the material. We specifically consider as an example the following problem.

4.1.1 The Diffusion PDE

$$\begin{aligned} u_t = u_{xx}, &\quad 0 \leq x \leq 1, t \geq 0 \\ u(0,t) = 0 = u(1,t), &\quad t > 0 \\ u(x,0) = f(x), &\quad 0 < x < 1 \end{aligned}$$

Let $u(x,t) = X(x)T(t)$, so that

$$\begin{aligned} &XT' = X''T \quad \text{for } 0 \leq x \leq 1, t \geq 0, \\ &X(0) = X(1) = 0. \end{aligned}$$

Now we obtain:

$$\frac{X''}{X} = \frac{T'}{T}$$

and we set this equal to a separation constant $-\lambda$ that will help us find a basis for the solutions.

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda \implies X'' = -\lambda X, \quad T' = -\lambda T.$$

Rearranging, we have:

$$X'' + \lambda X = 0 \quad T' + \lambda T = 0.$$

There are three cases for λ: zero, positive, negative.

Case 1: $\lambda = 0$. Then $X'' = 0$, which gives us $X = Ax + B$. The boundary conditions $X(0) = X(1) = 0$ imply that $B = 0$ and $A = 0$, which gives us $X = 0$.

Case 2: $\lambda < 0$. So $\lambda = -k^2$ for some $k > 0$. Then $X'' - k^2 X = 0$, which results in $X(x) = Ae^{kx} + Be^{-kx}$. However, the boundary conditions $X(0) = X(1) = 0$ imply that $A + B = 0$ and $Ae^k + Be^{-k} = 0$. This results in $A = -B$ so $A(e^k - e^{-k}) = 0$, and so $A = B = 0$. Thus again, $X \equiv 0$.

Case 3: $\lambda > 0$. So $\lambda = k^2$ for some $k > 0$. Then $X'' + k^2 X = 0$, which means $X(x) = A\cos(kx) + B\sin(kx)$. The boundary conditions $X(0) = X(1) = 0$ imply that $A = 0$ and $B\sin(k) = 0$. This time, we can get non-trivial solutions, when k is a multiple of π i.e. $k = n\pi$ for some $n \in \mathbb{Z}^+$
Thus we are interested in $\lambda = n^2\pi^2, X(x) = B\sin(n\pi x)$ for $n \in \mathbb{Z}^+$.
Now we deal with T. Since we know λ now, we have

$$T' + n^2\pi^2 T = 0$$

for some $n \in \mathbb{Z}^+$. We can solve this 1st order ODE:

$$T(t) = Ce^{-n^2\pi^2 t}.$$

So for each n, we combine $T(t)$ with $X(x)$ to get

$$u_n(x,t) = A_n e^{-n^2\pi^2 t} \sin n\pi x,$$

for some constant A_n. We then superimpose these solutions so,

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-n^2\pi^2 t} \sin n\pi x.$$

Finally, we can use the initial conditions, yielding

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x = f(x).$$

This is the half-range Fourier sine series of f, so

$$A_n = 2\int_0^1 f(x) \sin n\pi x \, dx.$$

If we were given an explicit f, we could evaluate this to get the final solution for u.

4.1.2 Wave Equation

Our second example of the application of Fourier series methods is to the partial differential equation describing a vibrating string, such as in a musical instrument like a piano.

Put $c = \sqrt{T_0/p}$ (which has the dimensions of length / time and is called the wave speed). Now suppose that the string is initially at rest with a known deflection $u_0(x)$, then

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= 0, && 0 < x < \ell, t > 0,\\
u(0,t) &= 0, && t > 0,\\
u(\ell,t) &= 0, && t > 0,\\
u(x,0) &= u_0(x), && 0 < x < \ell,\\
\frac{\partial u}{\partial t}(x,0) &= 0, && 0 < x < \ell.
\end{aligned}$$

Then the separation of variables technique used in the diffusion example follows almost exactly to solve for u in the wave equation.

4.2 Complete Orthogonal Systems

Expanding a function as a linear combination of orthogonal function leads naturally to the notion of a generalised Fourier series.

If $w : (a, b) \to \mathbb{R}$ satisfies

$$w(x) > 0 \quad \text{for } a < x < b,$$

then we define the inner product with **weight function** w by

$$\langle f, g\rangle_w = \langle f, gw\rangle = \int_a^b f(x)g(x)w(x)\,dx,$$

and the corresponding norm by

$$\|f\|_w = \sqrt{\langle f, f\rangle_w} = \sqrt{\int_a^b [f(x)]^2 w(x)\,dx}.$$

Two functions f and g are **orthogonal with respect to** w **over the interval** (a, b) if $\langle f, g\rangle_w = 0$.

A set of functions $S \subseteq L_2(a, b, w)$ is said to be **orthogonal** if every pair of functions in S is orthogonal and if no function is identically zero on (a, b).

We say that S is **orthonormal** if, in addition, each function has norm 1.

Orthogonal implies Independent If S is orthogonal then S is linearly independent.

Generalised Pythagorus Theorem If $\{\phi_1, \ldots, \phi_N\}$ is orthogonal then, for any $C_1, \ldots, C_N \in \mathbb{R}$,

$$\left\|\sum_{j=1}^{N} C_j\phi_j\right\|_w^2 = \sum_{j=1}^{N} C_j^2\|\phi_j\|_w^2.$$

Generalised Fourier Coefficients If f is in the span of an orthogonal set of functions $\{\phi_1, \phi_2, \ldots, \phi_N\}$ in $L_2(a, b, w)$, then the coefficients in the representation

$$f(x) = \sum_{j=1}^{N} A_j\phi_j(x)$$

are given by

$$A_j = \frac{\langle f, \phi_j\rangle_w}{\|\phi_j\|_w^2} \quad \text{for } 1 \le j \le N.$$

We call A_j the jth **Fourier coefficient** of f with respect to the given orthogonal set of functions.

Consider **approximating** a function $f \in L_2(a, b, w)$ by a function in the span of an orthogonal set $\{\phi_1, \phi_2, \ldots, \phi_N\}$, that is finding coefficients C_j such that

$$f(x) \approx \sum_{j=1}^{N} C_j\phi_j(x) \quad \text{for } a < x < b.$$

We seek to choose the C_j so that the **weighted mean-square error**

$$\left\| f - \sum_{j=1}^{N} C_j \phi_j \right\|_w^2 = \int_a^b \left(f(x) - \sum_{j=1}^{N} C_j \phi_j(x) \right)^2 w(x)\, dx$$

is as small as possible.

Least-Squares Approximation For all $C_1, C_2, \dots, C_N$, the weighted mean-square error satisfies

$$\left\| f - \sum_{j=1}^{N} C_j \phi_j \right\|_w^2 = \|f\|_w^2 - \sum_{j=1}^{N} A_j^2 \|\phi_j\|_w^2 + \sum_{j=1}^{N} (C_j - A_j)^2 \|\phi_j\|_w^2.$$

The Fourier coefficients satisfy Bessel's inequality, which is

$$\sum_{j=1}^{\infty} A_j^2 \|\phi_j\|_w^2 \leq \|f\|_w^2.$$

An orthogonal set S is **complete** if there is no non-trivial function in $L_2(a, b, w)$ orthogonal to every function in S, i.e. if the condition

$$\langle f, \phi \rangle_w = 0 \quad \text{for every } \phi \in S$$

implies that

$$\|f\|_w = 0.$$

In particular, if S is a complete orthogonal set, then every proper subset of S **fails** to be complete.

Example The set $S = \{\sin jx : j \geq 1 \text{ and } j \neq 7\}$ is **not** complete in $L_2(0, \pi)$ because $\sin 7x$ is orthogonal to every function in S.

Equivalent Definitions of Completeness If $S = \{\phi_1, \phi_2, \dots\}$ is orthogonal in $L_2(a, b, w)$, then the following properties are equivalent:

1. S is complete;
2. for each $f \in L_2(a, b, w)$ if A_j denotes the jth Fourier coefficient of f then

$$\left\| f - \sum_{j=1}^{N} A_j \phi_j \right\|_w \to 0 \text{ as } N \to \infty;$$

3. each function $f \in L_2(a, b, w)$ satisfies **Parseval's identity**:

$$\|f\|_w^2 = \sum_{j=1}^{\infty} A_j^2 \|\phi_j\|_w^2.$$

Least-squares Error If $S = \{\phi_1, \phi_2, \phi_3, \dots\}$ is a complete orthogonal sequence in $L_2(a, b, w)$, then for any $f \in L_2(a, b, w)$,

$$\|e_N\|^2 = \sum_{j=N+1}^{\infty} A_j \|\phi_j\|_w^2.$$

4.3 Sturm-Liouville Problems

An ODE of the form

$$[p(x)u']' + [\lambda r(x) - q(x)]u = 0, \quad a < x < b,$$

is called a **Sturm-Liouville** equation. The coefficients p, q, r must all be real-valued with

$$p(x) > 0 \text{ and } r > 0 \text{ for } a < x < b.$$

Defining the formally self-adjoint differential operator

$$Lu = -[p(x)u']' + q(x)u,$$

we can write the ODE as

$$Lu = \lambda r u \quad \text{ on } (a, b).$$

Any non-trivial (possibly complex-valued) solution u satisfying $Lu = \lambda r u$ on (a, b) (plus appropriate boundary conditions) is said to be an **eigenfunction** of L with **eigenvalue** λ. In this case, we refer to (ϕ, λ) as an **eigenpair**.

Legendre's Equation Legendre's equation

$$(1 - x^2)u'' - 2xu' + \nu(\nu + 1)u = 0$$

is equivalent to

$$[(1 - x^2)u']' + \nu(\nu + 1)u = 0$$

which is of the Sturm-Liouville form

$$p(x) = 1 - x^2, \quad q(x) = 0, \quad r(x) = 1, \quad \lambda = \nu(\nu + 1).$$

Assume as before that p, q, r are real-valued with $p(x) > 0$ and $r(x) > 0$ for $a < x < b$. A **regular Sturm-Liouville eigenproblem** is of the form

$$\begin{aligned} Lu &= \lambda r u && \text{for } a < x < b, \\ B_1 u = b_{11}u' + b_{10}u &= 0 && \text{at } x = a, \\ B_2 u = b_{21}u' + b_{20}u &= 0 && \text{at } x = b. \end{aligned}$$

where a and b are finite with

$$p(a) \neq 0 \text{ and } p(b) \neq 0,$$

and where $b_{10}, b_{11}, b_{20}, b_{21}$ are real with

$$|b_{10}| + |b_{11}| \neq 0 \text{ and } |b_{20}| + |b_{21}| \neq 0.$$

Eigenfunctions are Orthogonal Let L be a Sturm-Liouville differential operator. If $u, v : [a, b] \to \mathbb{C}$ satisfy

$$Lu = \lambda r u \text{ on } (a, b), \quad \text{with } B_1 u = 0 = B_2 u,$$

and

$$Lv = \mu r v \text{ on } (a, b), \quad \text{with } B_1 v = 0 = B_2 v,$$

and if $\lambda \neq \mu$, then u and v are orthogonal on the interval (a, b) with respect to the weight function $r(x)$, i.e.,

$$\langle u, v \rangle_r = \int_a^b u(x) v(x) r(x) \, dx = 0.$$

Eigenvalues are Real Let L be a Sturm-Liouville differential operation. If $u : [a, b] \to \mathbb{C}$ is not identically zero and satisfies

$$Lu = \lambda r u \text{ on } (a, b), \quad \text{with } B_1 u = 0 = B_2 u,$$

then λ is real.

Completeness of the Eigenfunctions The regular Sturm-Liouville problem has a infinite sequence of eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$ with corresponding eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots$ and moreover:

1. the eigenfunctions $\phi_1, \phi_2, \phi_3, \ldots$ form a complete orthogonal system on the interval (a, b) with respect to the weight function $r(x)$;
2. the eigenvalues satisfy $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ with $\lambda_j \to \infty$ as $j \to \infty$.

4.4 Elliptic Differential Operators

We now return to the study of PDEs by briefly introducing some concepts related to a more general study of second-order PDEs - namely, ellipticity and divergence form PDEs.

Vector Calculus Notation Partial derivative operator $\partial_j = \partial / \partial x_j$.
For a scalar field $u : \mathbb{R}^d \to \mathbb{R}$, the **gradient** is the vector field $\operatorname{grad} u : \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\operatorname{grad} u = \boldsymbol{\nabla} u = \sum_{j=1}^{d} \partial_j u \mathbf{e}_j = \begin{bmatrix} \partial_1 u \\ \partial_2 u \\ \cdots \\ \partial_d u \end{bmatrix}$$

For a vector field $\mathbf{F} : \mathbb{R}^d \to \mathbb{R}^d$, the **divergence** is the scalar field $\operatorname{div} \mathbf{F} : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\operatorname{div} \mathbf{F} = \boldsymbol{\nabla} \cdot \mathbf{F} = \sum_{j=1}^{d} \partial_j F_j = \partial_1 F_1 + \partial_2 F_2 + \cdots + \partial_d F_d.$$

Second-order Linear PDEs in $\mathbb{R}^d$ The most general second-order linear partial differential operator in $\mathbb{R}^d$ has the form

$$Lu = -\sum_{j=1}^{d} \sum_{k=1}^{d} a_{jk}(\mathbf{x}) \partial_j \partial_k u + \sum_{k=1}^{d} b_k(\mathbf{x}) \partial_k u + c(\mathbf{x}) u.$$

Laplacian The **Laplacian** is defined by $\boldsymbol{\nabla}^2 u = \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} u) = \operatorname{div}(\operatorname{grad} u)$, that is,

$$\boldsymbol{\nabla}^2 u = \sum_{j=1}^{d} \partial_j^2 u = \partial_1^2 u + \partial_2^2 u + \cdots + \partial_d^2 u.$$

Thus, $-\boldsymbol{\nabla}^2 u$ has the form of the second-order linear PDE with

$$a_{jk}(\mathbf{x}) = \delta_{jk}, \quad b_k(\mathbf{x}) = 0, \quad c(\mathbf{x}) = 0.$$

We call

$$L_0 u = \sum_{j=1}^{d} \sum_{k=1}^{d} a_{jk}(\mathbf{x}) \partial_j \partial_k u$$

the **principal part** of the partial differential operator.

A second-order linear partial different operator is uniformly **elliptic** in a subset $\omega \subseteq \mathbb{R}^d$ if there exists a positive constant c such that

$$\xi^{\mathbf{T}} A(\mathbf{a}) \xi \geq c \|\xi\|^2 \quad \text{for all } \mathbf{a} \in \Omega \text{ and } \xi \in \mathbb{R}^d.$$

Elliptic The operator $L = -\boldsymbol{\nabla}^2$ is elliptic (with $c = 1$) on any $\Omega \subseteq \mathbb{R}^d$, since

$$\sum_{j=1}^{d} \sum_{k=1}^{d} \delta_{jk} \xi_j \xi_k = \sum_{k=1}^{d} \xi_k^2 = \|\xi\|^2.$$

Not Elliptic The operator $L = -(\partial_1^2 + 2\partial_2^2 - \partial_3^2)$ is **not** elliptic in $\mathbb{R}^3$ since in this case the quadratic form

$$\xi^T A \xi = \begin{bmatrix} \xi_1 & \xi_2 & \xi_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \xi_1^2 + 2\xi_2^2 - \xi_3^2$$

is negative if $\xi_1 = \xi_2 = 0$ and $\xi_3 \neq 0$.

Symmetry and Skew-Symmetry Put

$$\begin{aligned} a_{jk}^{sy} &= \frac{1}{2}(a_{jk} + a_{kj}) = \text{ symmetric part of } a_{jk} \\ a_{jk}^{sk} &= \frac{1}{2}(a_{jk} - a_{kj}) = \text{ skew-symmetric part of } a_{jk}, \end{aligned}$$

so that

$$a_{jk} = a_{jk}^{sy} + a_{jk}^{sk}, \quad a_{kj}^{sy} = a_{jk}^{sy}, \quad a_{kj}^{sk} = -a_{jk}^{sk}$$

When investigating if L is elliptic, it suffices to look at a_{jk}^{sy}.

Lemma

$$\sum_{j=1}^{d} \sum_{k=1}^{d} a_{jk}(\mathbf{x}) \xi_j \xi_k = \sum_{j=1}^{d} \sum_{k=1}^{d} a_{jk}^{sy}(\mathbf{x}) \xi_j \xi_k$$

Theorem Denote the eigenvalues of the real symmetric matrix $[a_{jk}^{sy}]$ by $\lambda_j(x)$ for $1 \le j \le d$. The operator of Second-order Linear PDEs is elliptic on Ω if and only if t here exists a positive constant c such that

$$\lambda_j(\mathbf{x}) \ge c \quad \text{for } 1 \le j \le d \text{ and all } \mathbf{x} \in \Omega.$$

Elliptic

- $L = -(3\partial_1^2 + 2\partial_1\partial_2 + 2\partial_2^2)$ is elliptic.
- $L = -(\partial_1^2 - 4\partial_1\partial_2 + \partial_2^2)$ is not elliptic.

The Laplacian occurs in three of the most well studied PDEs:

1. **Poisson equation (Laplace's equation if $f \equiv 0$)** (elliptic):

$$-\boldsymbol{\nabla}^2 u = f.$$

2. **Diffusion equation** or **heat equation** (parabolic):

$$\frac{\partial u}{\partial t} - \boldsymbol{\nabla}^2 u = f.$$

3. **Wave equation** (hyperbolic):

$$\frac{\partial^2 u}{\partial t^2} - \boldsymbol{\nabla}^2 u = f.$$