Higher Linear Algebra
MATH2621 UNSW

Jeremy Le *

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Contents

*With some inspiration from Hussain Nawaz's Notes

1 Assumed Knowledge

- the definition of complex numbers,
- their arithmetic,
- Cartesian and polar representations,
- the Argand diagram,
- de Moivre's theorem, and
- extracting nth roots of complex numbers.

2 Inequalities and Sets of Complex Numbers

2.1 Equalities and Inequalities

Modulus Squared of a Sum For all complex numbers w and z,

$$|w+z|^2 = |w|^2 + 2\operatorname{Re}(w\bar{z}) + |z|^2.$$

Triangle Inequality For all complex numbers w and z,

$$|w+z| \leq |w| + |z| \qquad \forall w, z, \in \mathbb{C}.$$

Circle Inequality For all complex numbers w and z,

$$||w| - |z|| \leq |w - z|.$$

Modulus of e^z If $z \in \mathbb{C}$, then

$$|e^z| = e^{\operatorname{Re}(z)}.$$

Modolus of $e^z - 1$ inequality For all real numbers θ,

$$|e^{i\theta} - 1| \leq |\theta|.$$

2.2 Properties of Sets

Open Ball The open ball with centre z_0 and radius ϵ, written $\mathrm{B}(z_0, \epsilon)$, is the set $\{z \in \mathbb{C} : |z - z_0| < \epsilon\}$.

Punctured Open Ball The punctured open ball with centre z_0 and radius ϵ, written $B^\circ(z_0, \epsilon)$, is the set $\{z \in \mathbb{C} : 0 < |z - z_0| < \epsilon\}$.

Interior, Exterior and Boundary Points Suppose that $S \subseteq \mathbb{C}$. For any point z_0 in $\mathbb{C}$, there are three mutually exclusive and exhaustive possibilities:

(1) When the positive real number ϵ is sufficiently small, $\mathrm{B}(z_0, \epsilon)$ is a subset of S, that is, $\mathrm{B}(z_0, \epsilon) \cap S = \mathrm{B}(z_0, \epsilon)$. In this case, z_0 is an interior point of S.

(2) When the positive real number *epsilon* is sufficiently small, $\mathrm{B}(z_0, \epsilon)$ does not meet S, that is, $\mathrm{B}(z_0, \epsilon)\ capS = \emptyset$. In this case, z_0 is an exterior point of S.

(3) No matter how small the positive real number ϵ is, neither of the above holds, that is, $\emptyset \subset \mathrm{B}(z_0, \epsilon) \cap S \subset \mathrm{B}(z_0, \epsilon)$. In this case, z_0 is a boundary point of S.

Open, Closed, Closure, Bounded, Compact, Region Sets Suppose that $S \subseteq \mathbb{C}$.

(1) The set S is open if all its points are interior points.

(2) The set S is closed if it contains all of its boundary points, or equivalently, if its complement $\mathbb{C} \setminus S$ is open.

(3) The closure of the set S is the set consisting of the points of S together with the boundary points of S.

(4) The set is bounded if $S \subseteq \mathrm{B}(0, R)$ for some $R \in \mathbb{R}^+$

(5) The set S is compact if it is both closed and bounded.

(6) The set S is a region if it is an open set together with none, some, or all of its boundary points.

2.3 Arcs

Polygonal Arc A polygonal arc is a finite sequence of finite directed line segments, where the end point of one line segment is the initial point of the next one.

Simple Closed Polygonal Arc A simple closed polygonal arc is a polygonal arc that does not cross itself, but the final point of the last segment is the initial point of the first segment.

Interior and Exterior Arc The complement of a simple closed polygonal arc is made up of two pieces: one, the interior of the arc, is bounded, and the other, exterior is not.

Polygonally Path-connectedness Let $X \subseteq \mathbb{C}$ be a subset of the complex plane.

(1) The set X is polygonally path-connected if any two points of X can be joined by a polygonal arc lying inside X.

(2) The set X is simply polygonally connected if it is polygonally path-connected and if the interior of every simple closed polygonal arc in X lies in X, that is, if "X has no holes".

(3) The set X is a domain if it is open and polygonally path-connected.

3 Functions of a Complex Variable

Complex Function A complex function is one whose domain, or whose range, or both, is a subset of the complex plane $\mathbb{C}$ that is not a subset of the real line $\mathbb{R}$.

Complex Polynomial A complex polynomial is a function $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = a_d z^d + \cdots + a_1 z + a_0,$$

where $a_d, \ldots, a_1, a_0 \in \mathbb{C}$. If $a_d \neq 0$, we say that p is of degree d. A rational function is a quotient of polynomials.

The Fundamental Theorem of Algebra Every nonconstant complex polynomial p of degree d factorizes: there exists $\alpha_1, \alpha_2, \ldots, \alpha_d$ and c in $\mathbb{C}$ such that

$$p(z) = c \prod_{j=1}^{d} (z - a_j).$$

Polynomial Division and Partial Fractions Suppose that p and q are polynomials. Then

$$\frac{p(z)}{q(z)} = s(z) + \frac{r(z)}{q(z)},$$

where r and s are polynomials, and the degree of r is strictly less than the degree of q. Further, if

$$q(z) = c \prod_{j=1}^{e} (z - \beta_j)^{m_j},$$

then we may decompose the term r/q into partial fractions:

$$\frac{r(z)}{q(z)} = \sum_{j=1}^{e} \sum_{k=1}^{m_j} \frac{a_{jk}}{(z - \beta_j)^k}.$$

Real and Imaginary Parts To a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$, we associate two real-valued functions u and v of two real variables:

$$f(x + iy) = u(x, y) + iv(x, y).$$

Then $u(x, y) = \operatorname{Re} f(x + iy)$ and $v(x, y) = \operatorname{Im} f(x + iy)$.

3.1 The function $w = 1/z$

Consider the mapping $w = 1/z$.

(1) The image of a line through 0 (with the origin removed) is a line through 0 (with the origin removed).

(2) The image of a line that does not pass through 0 is a circle (with the origin removed). If p is the closest point on the line to 0, then the line segment between 0 and 1/p is a diameter of the circle.

(3) The image of a circle that passes through 0 is a line. If q is the furthest point on the circle from 0, then the closest point on the line to 0 is $1/q$.

(4) The image of a circle that does not pass through 0 is a circle. If p and q are the closest and furthest point on the circle from 0, then the closest and furthest point on the image circle to 0 are $1/q$ and $1/p$.

3.2 Fractional Linear Transformations

Factorising Matrices Every 2×2 complex matrix with determinant 1 may be written as a product f at most three matrices of the following special types:

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Image of Lines and Circles Let T_M be a fractional linear transformation. Then the image of a line under T_M is a line or a circle, and the image of a circle under T_M is also a line or a circle.

4 Limits and Continuity

4.1 Limits

Definition of a Limit Suppose that f is a complex function and that z_0 is in $\text{Domain}(f)^-$. We say that $f(z)$ tends to ℓ as z tends to z_0, or that ℓ is the limit of $f(z)$ as z tends to z_0, and we write $f(z) \to \ell$ as $z \to z_0$, or

$$\lim_{z \to z_0} f(z) = \ell,$$

if, for every $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(z) - \ell| < \epsilon$ provided that z is in $\text{Domain}(f)$ and $0 < |z - z_0| < \delta$.

Limit within a Subset Suppose also S is a subset of $\text{Domain}(f)$ and that $z_0 \in \bar{S}$. We say that $f(z)$ tends to ℓ as z tends to z_0 in S, or that ℓ is the limit of $f(z)$ as z tends to z_0 in S, and write $f(z) \to \ell$ as $z \to z_0$ in S, or

$$\lim_{\substack{z \to z_0 \\ z \in S}} f(z) = \ell,$$

if, for every $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $|f(z) - \ell| < \epsilon$ provided that $z \in S$ and $0 < |z - z_0| < \delta$.

Limits at Infinity Suppose that f is a complex function, that $\ell \in \mathbb{C} \subset \{\infty\}$, and that either $z_0 \in \mathrm{Domain}(f)^-$ or $Domain(f)$ is not bounded and $z_0 = \infty$. We say that $f(z)$ tends to ℓ as z tends to z_0, or that ℓ is the limit of $f(z)$ as z tends to z_0, and we write $f(z) \to \ell$ as $z \to z_0$, or

$$\lim_{z\to z_0} f(z) = \ell,$$

if for all $\epsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that $f(z) \in \mathrm{B}(\ell, \epsilon)$ provided that $z \in \mathrm{B}^\circ(z_0, \delta)$.

Standard Limits Suppose that $\alpha, c \in \mathbb{C}$. Then

$$\lim_{z\to\alpha} c = c \qquad \lim_{z\to\infty} c = c$$

$$\lim_{z\to\alpha} z - c = \alpha - c \qquad \lim_{z\to\infty} z - \alpha = \infty$$

$$\lim_{z\to\alpha} \frac{1}{z-\alpha} = \infty \qquad \lim_{z\to\alpha} \frac{1}{z-\alpha} = 0$$

Lemmas on Limits

1. Suppose that f is a complex function, that $T \subseteq S \subseteq \mathrm{Domain}(f)$, and that $z_0 \in \bar{T}$. If $\lim_{\substack{z\to z_0 \\ z\in S}} f(z)$ exists, then so does $\lim_{\substack{z\to z_0 \\ z\in T}} f(z)$, and they are equal.
2. Suppose that f is a complex function, and that $z_0 \in \mathrm{Domain}(f)^-$. If $\lim_{z\to z_0} f(z)$ exists, then it is unique.

Algebra of Limits Suppose that f and g are complex functions and that $c \in \mathbb{C}$. Then

$$\lim_{z\to z_0} cf(z) = c \lim_{z\to z_0} f(z)$$

$$\lim_{z\to z_0} f(z) + g(z) = \lim_{z\to z_0} f(z) + \lim_{z\to z_0} g(z)$$

$$\lim_{z\to z_0} f(z)g(z) = \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z)$$

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)},$$

in the sense that if the right hand side exists, then so does the left hand size and they are equal. In particular, for the quotient, we require that $\lim_{z\to z_0} g(z) \neq 0$.

Limits and Complex Conjugation Suppose that f is a complex function and that either $\mathrm{Domain}(f)$ is unbounded and $z_0 = \infty$ or $z_0 \in \mathrm{Domain}(f)^-$. Then

$$\lim_{z\to z_0} \overline{f(z)} = \overline{\lim_{z\to z_0}} f(z)$$

$$\lim_{z\to z_0} \mathrm{Re}(f(z)) = \mathrm{Re} \lim_{z\to z_0} f(z)$$

$$\lim_{z\to z_0} \mathrm{Im}(f(z)) = \mathrm{Im} \lim_{z\to z_0} f(z)$$

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \mathrm{Re}(f(z)) + i \lim_{z\to z_0} \mathrm{Im}(f(z)),$$

in the sense that if the right hand side exists, then so does the left hand size, and they are equal. In particular, $f(z)$ tends to ℓ as z tends to z_0 if and only if $\mathrm{Re}(f(z))$ tends to $\mathrm{Re}(\ell)$ and $\mathrm{Im}(f(z))$ tends to $\mathrm{Im}(\ell)$ as z tends to z_0.

4.2 Continuity

Definition Suppose that the complex function f is defined in a set $S \subseteq \mathbb{C}$, and that $z_0 \in S$. We say that f is continuous at z_0 if

$$\lim_{z\to z_0} f(z) = f(z_0);$$

that is, the limit exists, $f(z_0)$ exists, and they are equal.

We say that f is continuous in S if it is continuous at all points of S, and continuous if it is continuous in its domain.

Properties of Continuous Functions

- Suppose that $c \in \mathbb{C}$, and that $f : S \to \mathbb{C}$ and $g : S \to \mathbb{C}$ are continuous complex functions in $S \subseteq \mathbb{C}$. Then $cf, f+g, |f|, \bar{f}, \mathrm{Re}\, f, \mathrm{Im}\, f$ and fg are continuous in S, as is f/g provided that $g(z) \neq 0$ for any z in S.
- Suppose that $f : S \to \mathbb{C}$ and $g : T \to \mathbb{C}$ are continuous complex functions in $S \subseteq \mathbb{C}$ and $T \subseteq \mathbb{C}$. Then $f \circ g$ is continuous where it is defined, that is, in $\{z \in T, g(z) \in S\}$.

Continuity and Boundedness Suppose that the set $S \subseteq \mathbb{C}$ is compact (i.e., closed and bounded) and that f is a continuous complex function defined on S. Then there exists a point z_0 in S such that

$$|f(z_0)| = \max\{|f(z)| : z \in S\}.$$

The Log Function The function $\mathrm{Log} : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is defined by

$$\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z).$$

5 Complex Differentiability

Definition Suppose that $S \subseteq \mathbb{C}$ and that $f : S \to \mathbb{C}$ is a complex function. Then we say that f is differentiable at the point z_0 in S if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \qquad \text{or equivalently} \qquad \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

exists. If it exists, it is called the derivative of f at z_0, and written $f'(z_0)$ or $\dfrac{df(z_0)}{dz}$.

5.1 The Cauchy-Riemann Equations

Definition Suppose that Ω is an open subset of $\mathbb{C}$, that f is a complex function defined in Ω, that $f(x+iy) = u(x,y) + iv(x,y)$, where u and v are real-valued functions of two real variables, adn that f is differentiable at $z_0 \in \Omega$. Then the partial derivative

$$\frac{\partial u}{\partial x}(x_0, y_0), \qquad \frac{\partial u}{\partial y}(x_0, y_0), \qquad \frac{\partial v}{\partial x}(x_0, y_0), \qquad \frac{\partial v}{\partial y}(x_0, y_0)$$

all exists, and

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0).$$

Further,

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

Differentiability by Cauchy-Riemann If the four partial derivatives $\partial u/\partial x, \partial v/\partial x, \partial u/\partial y$ and $\partial v/\partial y$ are all continuous in an open set Ω, then f is complex differentiable at $z_0 \in \Omega$ if and only if the Cauchy-Riemann equations hold at z_0, and if so, then

$$f'(x_0 + iy_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0).$$

5.2 Properties of the Derivative

Differentiability Implies Continuity Suppose that f is a complex function and that $z_0 \in \text{Domain}(f)$. If f is differentiable at z_0, then f is continuous at z_0.

Algebra of Derivatives Suppose that $z_0 \in \mathbb{C}$, that the complex functions f and g are differentiable at z_0, and that $c \in \mathbb{C}$. Then the functions $cf, f+g$ and fg are differentiable at z_0 and

$$\begin{aligned} (cf)'(z_0) &= cf'(z_0), \\ (f+g)'(z_0) &= f'(z_0) + g'(z_0), \\ (fg)'(z_0) &= f'(z_0)g(z_0) + f(z_0)g'(z_0). \end{aligned}$$

Further, if $g(z_0) \neq 0$, then the function f/g is differentiable at z_0, and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g(z_0)^2}.$$

Composed Functions Suppose that $z_0 \in \mathbb{C}$, that the complex function f is differentiable at $g(z_0)$, and that the complex function$g$ ios differentiable at z_0. Then the function $f \circ g$ is differentiable at z_0, and

$$(f \circ g)'(z_0) = f'(g(z_0))g'(z_0).$$

L'Hôpital's Rule Suppose that $z_0 \in \mathbb{C} \cup \{\infty\}$ and that the complex functions f and g are differentiable at z_0. If $\lim_{z \to z_0} f(z)/g(z)$ is indeterminate, that is, of the form $0/0$ or ∞/∞, and if $\lim_{z \to z_0} f'(z)/g'(z)$ exists, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}.$$

Consequences of the Cauchy-Riemann Equations Suppose that f is differentiable in a domain Ω in $\mathbb{C}$. Then

(a) if $f' = 0$ in Ω, then f is constant on Ω;

(b) if $|f|$ is constant, then f is constant on Ω;

(c) if $\mathrm{Re}(f)$ or $\mathrm{Im}(f)$ is constant, then f is constant on Ω.

Polar Coordinates Suppose that the complex function f is differentiable at the point $z_0 \in \mathbb{C} \setminus \{0\}$, and that $z_0 = r_0 e^{i\theta_0}$. Then

$$\frac{\partial u}{\partial \theta}(r_0, \theta_0) = -r_0 \frac{\partial v}{\partial r}(r_0, \theta_0) \qquad \text{and} \qquad \frac{\partial v}{\partial \theta}(r_0, \theta_0) = r_0 \frac{\partial u}{\partial r}(r_0, \theta_0).$$

Further,

$$\begin{aligned} f'(z_0) &= e^{-i\theta_0} \left(\frac{\partial u}{\partial r}(r_0, \theta_0) + i \frac{\partial v}{\partial r}(r_0, \theta_0) \right) \\ &= \frac{-ie^{-i\theta_0}}{r} \left(\frac{\partial u}{\partial \theta}(r_0, \theta_0) + i \frac{\partial v}{\partial \theta}(r_0, \theta_0) \right). \end{aligned}$$

Log is Differentiable The function Log is differentiable in $\mathbb{C} \setminus (-\infty, 0]$.

5.3 Inverse Functions

Differentiability of Inverse Functions Suppose that Ω and Υ are open subsets of $\mathbb{C}$, that $f : \Omega \to \Upsilon$ is one-to-one, and that $f(z_0) = w_0$. If f is differentiable at z_0 and f^{-1} is differentiable at w_0, then $(f^{-1})'(w_0) = 1/f'(z_0)$.

5.4 Differentiable Definition

Holomorphic Suppose that Ω is an open subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ is a function. If f is differentiable in Ω, that is, if it is differentiable at every point of Ω, then we say that f is holomorphic or (complex) analytic or regular in Ω, and we write $f \in H(\Omega)$.

Entire If $\Omega = \mathbb{C}$ and f is differentiable in Ω, then we say that f is entire.

6 Harmonic Functions

Harmonic Functions Suppose that $u : \Omega \to \mathbb{R}$ is a function, where Ω is an open subset of $\mathbb{R}^2$, and that u is twice continuously differentiable, that is, all the partial derivatives $\partial u/\partial x, \partial u/\partial y, \partial^2 u/\partial x^2, \partial^2 u/\partial x \partial y, \partial^2 u/\partial y \partial x$ and $\partial^2 u/\partial y^2$ exists and are continuous. Then we say that u is harmonic in Ω if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Finding Harmonic Functions Suppose that $f \in H(\Omega)$, where Ω is an open subset of $\mathbb{C}$, that f is twice continuously differentiable in Ω, and that

$$f(x + iy) = u(x, y) + iv(x, y)$$

in Ω, where u and v are real-valued. Then u and v are harmonic functions.

Existence of Harmonic Functions If Ω is a simply polygonally connected domain, and $u : \Omega \to \mathbb{R}$ is harmonic, then there exists a harmonic function $v : \Omega \to \mathbb{R}$ such that f, given by

$$f(x + iy) = u(x, y) + iv(x, y)$$

in Ω is holomorphic. Any two such functions v differ by an additive constant.

Harmonic Conjugate The function v is called a harmonic conjugate of u. The function f may often be determined using the fact that

$$f'(x + iy) = u_x(x, y) + iv_x(x, y) = u_x(x, y) - iu_y(x, y).$$

7 Power Series

Definition A (complex) power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where the centre z_0 and the coefficients a_n are all fixed complex numbers, and the variable z is complex. We take $(z - z_0)^0$ to be 1 for all z, even when $z = z_0$.

Radius of Convergence Every power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ has a radius of convergence ρ, given by the formulae

$$\rho = \left(\limsup_{n\to\infty} |a_n|^{1/n} \right)^{-1} = \left(\lim_{k\to\infty} \sup_{n\geq k} |a_n|^{1/n} \right)^{-1}.$$

The radius of convergence $\rho \in [0, +\infty]$ satisfies:

(a) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges if $|z-z_0| < \rho$

(b) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ does not converge if $|z-z_0| > \rho$

(c) $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ may converge for no, some or all z such that $|z-z_0| = \rho$.

The Ratio Test The radius of convergence is given by

$$\rho = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|},$$

as long as the limit exists or is $+\infty$.

The Root Test The radius of convergence is given by

$$\rho = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}}.$$

as long as the limit exists or is $+\infty$.

The Algebra of Power series Suppose that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ and $\sum_{n=0}^{\infty} b_n(z-z_0)^n$ converge in $B(z_0,\rho)$ to $f(z)$ and $g(z)$, and that $c \in \mathbb{C}$. Then the following series also converge in $B(z_0,\rho)$:

(a) $\sum_{n=0}^{\infty} ca_n(z-z_0)^n$, and its sum is $cf(z)$;

(b) $\sum_{n=0}^{\infty}(a_n+b_n)(z-z_0)^n$, and its sum is $f(z)+g(z)$;

(c) $\sum_{n=0}^{\infty} c_n(z-z_0)^n$, where $c_n = \sum_{j=0}^{n} a_j b_{n-j}$, and its sum is $f(z)g(z)$.

Power Series are Differentiable Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ in $B(z_0,\rho)$ and $\rho > 0$. Then f is differentiable in $B(z_0,\rho)$, and

$$f'(z) = \sum_{n=1}^{\infty} a_n n(z-z_0)^{n-1} = \sum_{m=0}^{\infty} a_{m+1}(m+1)(z-z_0)^m$$

in $B(z_0,\rho)$.

Repeatedly Differentiating Power Series Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ in $B(z_0,\rho)$. Then f may be differentiated as many times as desired, and

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\dots(n-k+1)a_n(z-z_0)^{n-k}.$$

In particular,

$$f^{(k)}(z_0) = k!a_k$$

. Further, the real valued functions u and v, such that $f(x+iy) = u(x,y)+iv(x,y)$, may be differentiated as many times as desired, and all their partial derivatives are continuous.

Power Series that Vanish on an Interval Suppose that $g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ in $B(z_0, \rho)$, and that $\epsilon > 0$. If $g(z_0 + t) = 0$ for all real t in $(-\epsilon, \epsilon)$, then $g(z) = 0$ for all z in $B(z_0, \rho)$.

Power Series that are Equal near the Centre Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and moreover that $g(z) = \sum_{n=0}^{\infty} b_n(z - z_0)^n$ in $B(z_0, \rho)$. If $f(z_0 + t) = g(z_0 + t)$ for all $t \in (-\epsilon, \epsilon)$, then $f(z) = g(z)$ for all $z \in B(z_0, \rho)$.

8 Exponential, Hyperbolic and Trigonometric Functions

8.1 The Exponential Function

Definition We define the exponential series by the formula

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \forall z \in \mathbb{C}.$$

Properties of the Exponential Series

1. $\exp(0) = 1$;
2. $\exp(z + w) = \exp(z)\exp(w)$ for all $z, w \in \mathbb{C}$;
3. $\exp(-z) = \exp(z)^{-1}$ for all $z \in \mathbb{C}$;
4. $\exp(z) \neq 0$ for all $z \in \mathbb{C}$;
5. $\exp'(z) = \exp(z)$ for all $z \in \mathbb{C}$;
6. if a function $f : \mathbb{C} \to \mathbb{C}$ satisfies $f(0) = 1$ and $f'(z) = f(z)$ for all $z \in \mathbb{C}$, then $f(z) = \exp(z)$ for all $z \in \mathbb{C}$;
7. $\exp(x + iy) = e^x(\cos(y) + i\sin(y))$ for all $x, y \in \mathbb{R}$.

Periodicity of the Exponential Mapping The exponential exp maps $\mathbb{C}$ onto $\mathbb{C} \setminus \{0\}$, and $\exp(z_1) = \exp(z_2)$ if and only if $z_1 - z_2 \in 2\pi i\mathbb{Z}$.

8.2 The Hyperbolic Functions

Definition We define the complex hyperbolic cosine and sine by the formulae

$$\cosh(z) = \frac{\exp(z) + \exp(-z)}{2} = \sum_{n\in\mathbb{N}} \frac{z^{2n}}{(2n)!}$$

and

$$\sinh(z) = \frac{\exp(z) - \exp(-z)}{2} = \sum_{n\in\mathbb{N}} \frac{z^{2n+1}}{(2n+1)!}$$

for all $z \in \mathbb{C}$.

Properties of the Hyperbolic Sine and Cosine

(i) $\cosh(-z) = \cosh(z)$

(ii) $\sinh(-z) = -\sinh(z)$

(iii) $\cosh'(z) = \sinh(z)$

(iv) $\sinh'(z) = \cosh(z)$

(v) $\cosh(z + 2\pi ik) = \cosh(z)$

(vi) $\sinh(z + 2\pi ik) = \sinh(z)$

(vii) $\cosh(z + w) = \cosh(z)\cosh(w) + \sinh(z)\sinh(w)$

(viii) $\sinh(z + w) = \sinh(z)\cosh(w) + \cosh(z)\sinh(w)$

(ix) $\cosh^2(z) - \sinh^2(z) = 1$

(x) $\cosh(x + iy) = \cosh(x)\cos(y) + i\sinh(y)\sin(y)$

(xi) $\sinh(x + iy) = \sinh(x)\cos(y) + i\cosh(x)\sin(y)$.

for all $w, z \in \mathbb{C}$, all $k \in \mathbb{Z}$ and all $x, y \in \mathbb{R}$.

8.3 The Trigonometric Functions

Definition We define the complex cosine and sine by the formulae

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} = \sum_{n\in\mathbb{N}} \frac{(-1)^n z^{2n}}{(2n)!}$$

and

$$\sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i} = \sum_{n\in\mathbb{N}} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

for all $z \in \mathbb{C}$.

9 Logarithms and Roots

Square Root We define the principle value of the square root as:

$$\mathrm{PV} w^{1/2} = \begin{cases} |w|^{1/2} e^{i\mathrm{Arg}(w)/2} & \text{if } w \neq 0 \\ 0 & \text{if } w = 0. \end{cases}$$

Logarithm Suppose that $w = e^z$ and $z = x + iy$. Then $w = e^x e^{iy}$, so

$$|w| = e^x \qquad \text{and} \qquad \text{Arg}w = \text{Arg}e^{iy}.$$

Then $x = \ln|w|$, and x is single-valued, but $y = \text{Arg}(w) + 2\pi k$, where $k \in \mathbb{Z}$; and y is multiple-valued. When $w \neq 0$, we write $z = \log(w)$ to indicate that z can be any one of the infinitely many complex numbers such that $e^z = w$ and we write $z = \text{Log}(w)$ to indicate the choice that $z = \ln|w| + i\text{Arg}(w)$.

nth Roots The principle value of the nth root is given by

$$\text{PV}z^{1/n} = \exp\left(\frac{\text{Log}(z)}{n}\right) = |z|^{1/n}e^{i\text{Arg}(z)/n}.$$

The function $\text{PV}z^{1/n}$ is differentiable in $\mathbb{C} \setminus (-\infty, 0]$.

10 Inverses of Exponential and Related Functions

10.1 The Exponential Function

Inverse of The Exponential Function The principle branch of the complex logarithm is the function Log from $\mathbb{C} \setminus \{0\}$ to $\mathbb{C}$, given by

$$\text{Log}(z) = \ln|z| + i\text{Arg}(z),$$

where $\text{Arg}(z)$ takes values in the range $(-\pi, \pi]$.

Differentiability For any branch $\log_\theta$ of the complex logarithm,

$$\log_\theta'(w) = \frac{1}{w}$$

for all $w \in \mathbb{C} \setminus R_\theta$.

10.2 Complex Powers

Definition Given $z \in \mathbb{C} \setminus \{0\}$ and $\alpha \in \mathbb{C}$, we define

$$z^\alpha = \exp(\alpha \log(z)).$$

The principle branch of z^α is found by using Log, the principle branch of the logarithm. That is, $\text{PV}z^\alpha = \exp(\alpha\text{Log}(z))$.

Differentiability of Complex Powers The function $z \mapsto \text{PV}z^\alpha$ is differentiable in $C \setminus (-\infty, 0]$, with derivative $\alpha\text{PV}z^\alpha/z$.

10.3 Inverse Hyperbolic Trigonometric Functions

Inverse Hyperbolic Sine The principal branch of the inverse hyperbolic sine function is given by

$$\mathrm{PV}\sinh^{-1} w = \mathrm{Log}(w + \mathrm{PV}(w^2+1)^{1/2}).$$

Differentiability of Inverse Hyperbolic Cosine The principle branch of the inverse hyperbolic sine function is differentiable in $\mathbb{C} \setminus ([i, +i\infty) \cup (-i\infty, -i])$. Further,

$$(\mathrm{PV}\sinh^{-1})'(w) = \frac{1}{\mathrm{PV}\sqrt{w^2+1}}.$$

Inverse Hyperbolic Cosine Similarly, we define

$$\mathrm{PV}\cosh^{-1}(w) = \mathrm{Log}(w + \mathrm{PV}(w+1)^{1/2}\mathrm{PV}(w-1)^{1/2}).$$

11 Contour Integrals

Curves Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a curve and

$$\gamma(t) = \gamma_1(t) + i\gamma_2(t),$$

where $\gamma_1, \gamma_2 : [a, b] \to \mathbb{R}$. Then we define

$$\gamma'(t) = \gamma_1'(t) + i\gamma_2'(t),$$

when both $\gamma_1'(t)$ and $\gamma_2'(t)$ exist. That is,

$$\mathrm{Re}(\gamma'(t)) = (\mathrm{Re}(\gamma))' \quad \text{and} \quad \mathrm{Im}(\gamma') = (\mathrm{Im}(\gamma))'.$$

Contour A contour is an oriented range of a piecewise smooth curve in the complex plane.

Integral of a Complex-Valued Function Suppose that $u, v : [a, b] \to \mathbb{R}$ are real-valued functions, and that $f : [a, b] \to \mathbb{C}$ is given by $f = u + iv$. We define

$$\int_a^b f(t)dt = \int_a^b (u(t) = iv(t))dt = \int_a^b u(t)dt + i\int_a^b v(t)dt,$$

provided that the two real integrals on the right hand side exist.

That is,

$$\mathrm{Re}\left(\int_a^b f(t)dt\right) = \int_a^b \mathrm{Re}(f(t))dt \quad \text{and} \quad \mathrm{Im}\left(\int_a^b f(t)dt\right) = \int_a^b \mathrm{Im}(f(t))dt.$$

Properties of Integration For $a, b, c, d \in \mathbb{R}$, $\lambda, \mu \in \mathbb{C}$, a real-valued differentiable function $h : [c, d] \to [a, b]$ such that $h(c) = a$ and $h(d) = b$, and complex-valued functions f and g.

$$\int_a^b \lambda f(t) + \mu g(t) dt = \lambda \int_a^b f(t)dt + \mu \int_a^b g(t)dt$$

$$\int_c^d f(h(t))h'(t)dt = \int_a^b f(s)ds$$

$$\int_a^b f'(t)g(t)dt = [f(b)g(b) - f(a)g(a)] - \int_a^b f(t)g'(t)dt$$

$$\int_a^b e^{\lambda t}dt = \left[\frac{e^{\lambda t}}{\lambda}\right]_{t=a}^{t=b} = \frac{e^{\lambda b} - e^{\lambda a}}{\lambda}$$

$$|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt.$$

Complex Line Integrals Given a (not necessarily simple) piecewise smooth curve $\gamma : [a, b] \to \mathbb{C}$ and a continuous (not necessarily differentiable) function f defined on the range of γ, we define the complex line integral $\int_\gamma f(z)dz$ by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt,$$

provide that the integral on the right hand side exists.

Properties of Complex Line Integrals Suppose that $\lambda, \mu \in \mathbb{C}$, that $\lambda : [a, b] \to \mathbb{C}$ is a piecewise smooth curve, and that f and g are complex functions defined on Range(λ). Then the following hold.

(a) The integral is linear:

$$\int_\gamma \lambda f(z) + \mu g(z)dz = \lambda \int_\gamma f(z)dz + \mu \int_\gamma g(z)dz.$$

(b) The integral is independent of parametrisation: if δ is a reparametrisation of γ that is also a piecewise smooth curve, then

$$\int_\lambda f(z)dz = \int_\delta f(z)dz.$$

(c) The integral is additive for joins: if $\gamma = \alpha \sqcup \beta$, then

$$\int_\gamma f(z)dz = \int_\alpha f(z)dz + \int_\beta f(z)dz.$$

(d) The integral depends on the orientation:

$$\int_{\gamma*} f(z)dz = -\int_\gamma f(z)dz.$$

(e) We may estimate the size of the integral:

$$|\int_\gamma f(z)dz| \leq ML,$$

where L is the length of γ and M is a number such that $|f(z)| \leq M$ for all $z \in \text{Range}(\gamma)$.

Contour Integrals We define

$$\int_\Gamma f(z)dz = \int_\gamma f(z)dz,$$

where γ is any parametrisation of Γ.

12 The Cauchy-Goursat Theorem

12.1 Simply Connected Domains

The Cauchy-Goursat Theorem Suppose that Ω is a simply connected domain, that $f \in H(\Omega)$, and that Γ is a closed contour in Ω. Then

$$\int_\Gamma f(z)dz = 0.$$

Independence of Contour Suppose that Ω is a simply connected domain in $\mathbb{C}$, that $f \in H(\Omega)$, and that Γ and Δ are contours with the same initial point p and the same final point q. Then

$$\int_\Gamma f(z)dz = \int_\Delta f(z)dz.$$

Existence of Primitives Suppose that Ω is a simply connected domain in $\mathbb{C}$, and that $f \in H(\Omega)$. Then there exists a function F on Ω such that

$$\int_\Gamma f(z)dz = F(q) - F(p)$$

for all simple contours Γ in Ω from p to q. Further, F is differentiable, and $F' = f$. Finally, if F_1 is any other function such that $F_1' = f$, then $F_1 - F$ is a constant and

$$\int_\Gamma f(z)dz = F_1(q) - F_1(p),$$

where p and q are the initial and final points of Γ.

12.2 Multiply Connected Domains

Cauchy-Goursat Suppose that Ω is a bounded domain whose boundary $\partial\Omega$ consists of finitely many contours, $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$. Suppose also that $f \in H(\Upsilon)$, where $\bar{\Omega} \subset \Upsilon$. Then

$$\int_\partial \Omega f(z)dz = \sum_{i=0}^{n} \int_{\Gamma_j} f(z)dz = 0.$$

Corollary Suppose that Υ is a simply connected domain, that Γ is a simple closed contour in Υ, and that f is a differentiable function in Υ. Then

$$\int_\Gamma f(z)dz = 0.$$

Existence of Primitives Suppose that Ω is a bounded domain whose boundary $\partial\Omega$ consists of finitely many contours $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$, that $\bar{\Omega} \subset \Upsilon$, and that f is a differentiable function in Υ. If $\int_{\Gamma_j} f(z)dz = 0$ when $j = 1, \ldots, n$, then $\int_\Gamma f(z)dz = 0$ for any closed contour in Ω, and further, there is a differentiable function F in Ω such that $F' = f$ and

$$\int_\Delta f(z)dz = F(q) - F(p)$$

for all simple contours Δ in Ω from p to q.

13 Cauchy's Integral Formula

Cauchy's Integral Formula Suppose that Ω is a simply connected domain in $\mathbb{C}$, that $f \in H(\Omega)$, that Γ is a simple closed contour in Ω and that $w \in \mathrm{Int}(\Gamma)$. Then

$$f(w) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z-w} dz.$$

Independence of Contour Suppose that w lies in a simply connected domain Ω, and that $f \in H(\Omega)$. If *Gamma* and Δ are simple closed contours such that $w \in \mathrm{Int}(\Gamma)$ and $w \in \mathrm{Int}(\Delta)$, then

$$\int_\Gamma \frac{f(z)}{z-w} dz = \int_\Delta \frac{f(z)}{z-w} dz.$$

Mean Value Formula Suppose that Ω is a simply connected domain in $\mathbb{C}$, that $f \in H(\Omega)$, and that $w \in \Omega$. If $\bar{B}(w,r) \subset \Omega$, then

$$f(w) = \frac{1}{2\pi} \int_0^{2\pi} f(w + re^{i\theta}) d\theta.$$

Cauchy's Generalised Integral Formula Suppose that $f \in H(B(z_0, R))$, and that Γ is a simple closed contour in $B(z_0, R)$ such that $z_0 \in \mathrm{Int}(\Gamma)$. Then

$$f(w) = \sum_{n=0}^{\infty} c_n (w - z_0)^n \qquad \forall w \in B(z_0, R),$$

where

$$c_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

The radius of convergence of the power series is at least R.

This combined with the fact that $f^{(n)}(z_0) = n! c_n$, implies that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

Liouville's Theorem Suppose that f is a bounded entire function. Then f is constant.

The Fundamental Theorem of Algebra Suppose that f is a nonconstant complex polynomial. Then f has at least one root, and hence f may be factorised as a product of a constant and finitely many linear factors.

Holomorphic Function Near a Zero Suppose that $f(z) = \sum_{n=0}^{\infty} a_n(z-w)^n$ for all $z \in B(w,r)$, and that $a_n \neq 0$ for some $n \in \mathbb{N}$. Let $N = \min\{n \in \mathbb{N} : a_n \neq 0\}$. Then

$$\lim_{z \to w} \frac{f(z)}{a_N(z-w)^N} = 1.$$

Zeros of a Holomorphic Funciton are Isolated Suppose that Ω is an open set, that $f \in H(\Omega)$, and that $f(w) = 0$ for some $w \in \Omega$. Then there exists $r \in \mathbb{R}^+$ such that either $f(z) = 0$ for all $z \in B(w,r)$ or $f(z) \neq 0$ for all $z \in B^\circ(w,r)$.

14 Morera's Theorem and Analytic Continuation

14.1 Morera's Theorem

Morera's Theorem Suppose that Ω is a domain, that hte function $f : \Omega \to \mathbb{C}$ is continuous, and that

$$\int_A f(z)dz = \int_B f(z)dz,$$

whenever the simple contours A and B have the same initial point and the same final point. Then f is holomorphic in Ω.

Holomorphic Extension Suppose that Λ is a (possibly infinite) line segment in an open set Ω and $\Omega \setminus \Lambda$ is open. If function $f : \Omega \to \mathbb{C}$ is continuous in Ω and is holomorphic in $\Omega \setminus \Lambda$, then f is holomorphic in Ω.

14.2 Analytic Continuation

f is 0 for Ball in Ball Suppose that $B(z_1, r_1) \subset B(w,r)$, that $f \in H(B(w,R))$, and that $f(z) = 0$ for all $z \in B(z_1, r_1)$. Then $f(z) = 0$ for all $z \in B(w,R)$.

Theorem Suppose that Υ is a nonempty open subset of a domain Ω in $\mathbb{C}$, and that $f \in H(\Omega)$. If $f(z) = 0$ for all z in Υ, then $f(z) = 0$ for all z in Ω.

Corollary Suppose that Υ is a nonempty open subset of a domain Ω in $\mathbb{C}$, and that $f, g \in H(\Omega)$. If $f(z) = g(z)$ for all z in Υ, then $f(z) = g(z)$ for all z in Ω.